Yamabe type equations with sign-changing nonlinearities on non-compact Riemannian manifolds

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Abstract

[1] In this work, we study the existence problem for positive solutions of the Yamabe type equation

$$\Delta u + q(x)u - b(x)u^\sigma = 0, \qquad \sigma > 1, \tag{Y}$$

on complete manifolds possessing a pole, the main novelty being that $b(x)$ is allowed to change signs. This relevant class of PDEs arises in a number of different geometric situations, notably the (generalized) Yamabe problem, but the sign-changing case has remained basically unsolved in the literature, with the exception of few special cases. This paper aims at giving a unified treatment for (Y), together with new, general existence theorems expressed in terms of the growth of $|b(x)|$ at infinity with respect to the geometry of the manifold and to $q(x)$. We prove that our results are sharp and that, even for $\mathbb{R}^m$, they improve on previous works in the literature. Furthermore, we also detect the asymptotic profile of $u(x)$ as x diverges, and a detailed description of the influence of $q(x)$ and of the geometry of M on this profile is given. The possibility to express the assumptions in an effective and simple way also depends on some new asymptotic estimates for solutions of the linear Cauchy problem $(vh')' + Avh = 0$, $h(0) = 1$, $h'(0) = 0$, of independent interest.

Introduction

Let $(M, \langle\, , \rangle)$ be a connected, non-compact Riemannian manifold of dimension $m \geq 2$, and let $q(x), b(x) \in C^\infty(M)$, $\sigma > 1$. In recent years, the Yamabe type equation

$$\Delta u + q(x)u - b(x)u^\sigma = 0 \qquad \text{on } M \tag{1}$$

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has been an important subject of investigation for many authors. One of the main geometric motivations comes from the (generalized) Yamabe problem, that is, that of finding a metric $\widetilde{\langle\,,\,\rangle}$, conformally related to $\langle\,,\,\rangle$, and with assigned scalar curvature $\widetilde{s}(x)$. When $m \geq 3$ and writing the conformal deformation as

$$\widetilde{\langle\,,\,\rangle} = u^{\frac{4}{m-2}}\langle\,,\,\rangle, \qquad \text{for some } u \in C^\infty(M),\ u > 0, \tag{2}$$

the relation between the background scalar curvature $s(x)$ and $\widetilde{s}(x)$ (first considered from an analytical point of view in [36]) is expressed by the equation

$$\Delta u - \frac{s(x)}{c_m}u + \frac{\widetilde{s}(x)}{c_m}u^{\frac{m+2}{m-2}} = 0, \tag{3}$$

where $c_m = \frac{4(m-1)}{m-2}$ and Δ is the Laplace-Beltrami operator in the metric $\langle\,,\,\rangle$. Thus, the original geometric task translates into that of the existence of a positive smooth solution u of (3). The case where $\widetilde{s}(x)$ (more generally, $b(x)$ in (1)) is allowed to change signs reveals to be the most challenging, and very little is known either about the existence or the non-existence of positive solutions.

In this paper, we shall study in detail this problem on manifolds $(M, \langle\,,\,\rangle)$ possessing a pole o, and whose radial sectional curvature with respect to o satisfies either

$$K_{\mathrm{rad}}(x) \leq -G\big(r(x)\big) \qquad r(x) = \mathrm{dist}(x, o) \tag{4}$$

or the two-sided bound

$$-\bar{G}\big(r(x)\big) \leq K_{\mathrm{rad}}(x) \leq -G\big(r(x)\big), \tag{5}$$

for some continuous functions $G, \bar{G}$. We recall that $K_{\mathrm{rad}}(x)$ is defined as the restriction of the sectional curvature to the 2-planes at x containing $\nabla r(x)$. As we shall see, G and $\bar{G}$ will be subjected to mild assumptions: eventually, the sole requirement we need on G is that the model manifold M_g, constructed from G and to which M is compared, be non-parabolic. Therefore, the class of manifolds to which our techniques apply is large (for instance, it includes every Cartán-Hadamard manifold of dimension $m \geq 3$, as well as hyperbolic spaces of any dimension), and enables us to formulate a sound judgement on the influence of geometry on the problem. In particular, the critical growth of $b(x)$ will naturally appear, and the reasons for its criticality will become evident. Specializing to manifolds close, in a broad sense, to the hyperbolic space, our results reveal quite appealing, and show in full strength their sharpness; for instance, see Theorem 2 below.

In the case when $\sigma \leq (m+2)/(m-2)$ and $b(x) < 0$ somewhere, the existence problem for (1) has often been studied via a combination of concentration-compactness methods and variants of the mountain pass theorem inspired by the seminal paper [4]. Among the literature we limit ourselves to quote the work of Q.S. Zhang, [37], for both a sharp result and an up-to-date account on the problem, suggesting the interested reader to consult the references therein for further insight. As in all the variational approaches to the non-compact Yamabe problem we are aware of, the method in [37] requires the validity of a global Sobolev-Poincaré type inequality on M via the positivity of a Yamabe type invariant, and the non-positivity of the term $q(x)$ in (1). Furthermore, to obtain uniform L^∞ estimates one has to assume that the volume growth of geodesic balls is at most Euclidean. Although the method works at its best in the Euclidean setting, on general non-compact manifolds the combination of the above requirements seem quite demanding, if only because they exclude manifolds close to the hyperbolic space. For such manifolds, to the best of our knowledge, the existence problem for (1) with sign-changing $b(x)$ is basically unsolved. In this paper, we follow a different approach via radialization arguments and the monotone iteration scheme, inspired by a pioneering work of W.M. Ni. In the Euclidean setting, and for $q(x) = 0$, in [24] Ni has given optimal conditions for the existence of positive solutions of (1) in terms of the growth of $|b(x)|$, and in the subsequent joint paper with W.Y. Ding, [6], they describe a whole variety of phenomena to illustrate how subtle is the dependence of u upon the behaviour of $b(x)$. Ni's result have subsequently been refined and extended by M. Naito, [23] and N. Kawano, [12]. Summarizing, they have proved the following

Theorem 1 ([24], Theorem 1.4, and [23], [12]). *Consider the Euclidean space $\mathbb{R}^m$, $m \geq 3$, and let $\widetilde{s}(x) \in C^\infty(\mathbb{R}^m)$ be a function satisfying*

$$|\widetilde{s}(x)| \leq B(r(x)), \tag{6}$$

for some function $B \in C^0(\mathbb{R}^+_0)$ such that

$$rB(r) \in L^1(+\infty). \tag{7}$$

Then, for each $\Gamma_2 > 0$ and sufficiently small, the Euclidean metric $\langle\, ,\, \rangle$ can be conformally deformed to a complete, smooth metric $\widetilde{\langle\, ,\, \rangle}$ of scalar curvature $\widetilde{s}(x)$ and satisfying

$$\Gamma_1\langle\, ,\, \rangle_x \leq \widetilde{\langle\, ,\, \rangle}_x \leq \Gamma_2\langle\, ,\, \rangle_x \qquad \forall\, x \in \mathbb{R}^m, \tag{8}$$

for some $0 < \Gamma_1 \leq \Gamma_2$, and

$$\widetilde{\langle\, ,\, \rangle}_x \to C\langle\, ,\, \rangle_x \qquad \text{as} \quad r(x) \to +\infty, \tag{9}$$

for some constant $C \in [\Gamma_1, \Gamma_2]$.

Remark 1. Ni stated the theorem under the requirement

$$|\widetilde{s}(x)| \leq \frac{C}{r(x)^l}, \tag{10}$$

for some $C > 0$ and $l > 2$, but he himself observed, already in [24], that

$$|\widetilde{s}(x)| \leq \frac{C}{r(x)^2 \log^2 r(x)} \qquad \text{for } r(x) >> 1. \tag{11}$$

is sufficient.

Remark 2. The case $C = 0$ in (9), that is, when the conformal factor $u \to 0$ as $r(x) \to +\infty$, reveals to be subtle, and the sole (7) is not sufficient to ensure the existence of a positive u decaying to 0, as it has been shown in [18].

Condition (7) is essentially sharp. In fact, by Theorem 1.13' of [24] or Theorem A of [2], no conformal deformation of $\mathbb{R}^m$ exists whenever the new scalar curvature $\widetilde{s}(x)$ is required to satisfy

$$\widetilde{s}(x) \leq 0 \quad \text{on } \mathbb{R}^m, \qquad \widetilde{s}(x) \leq -\frac{C}{r(x)^2 \log r(x)} \qquad \text{for } r(x) >> 1. \tag{12}$$

for some constant $C > 0$.

Remark 3. By virtue of Theorem 1, one could expect that the condition

$$\widetilde{s}(x) \leq -B(r(x)) \leq 0 \qquad \text{and} \qquad rB(r) \notin L^1(+\infty) \tag{13}$$

prevents the existence of any positive solution of $\Delta u = -\widetilde{s}(x)u^\sigma$ on $\mathbb{R}^m$. This claim is stated as a conjecture in [5] and, to the best of our knowledge, remains an open problem. In [19], [15] and [5], a number of steps have been moved towards the solution of this conjecture, giving rise to non-existence conditions slightly more demanding than (13), see in particular Theorems 2.2 and 2.3 in [5].

Investigating the generalized Yamabe problem on manifolds different from Euclidean space (with particular emphasis on the hyperbolic space), forces us to face three different kinds of problems. Firstly, these manifolds are (usually) not scalar flat, hence the term $q(x)$ in (1) must be taken into account and makes the matter more delicate. Secondly, in the course of their proofs Ni and Naito have exploited some ad-hoc functions that, for spaces other than $\mathbb{R}^m$, appear to be extremely difficult to construct. Thirdly, it is not clear whether Ni's radialization techniques can be coupled with standard comparison results for the Laplacian in order to obtain conclusions on manifolds which are not radially symmetric. Therefore, although Ni's approach looks promising, his method is insufficient to deal with the general case, and calls for the use of new techniques. This is the starting point of the present work. We remark that one of the main difficulties in the study of (1) is that, since $b(x)$ is allowed to change signs, we cannot use comparison arguments. Therefore, a challenging task will be to make sure that the subsolution we construct lies below the supersolution, in order for the monotone iteration scheme to be applicable. As we shall see, this problem will be made much more difficult due to the presence of nonradial $q(x)$.

Via the Laplacian comparison theorem, under assumptions (4) or (5) the Laplacian of the distance function $r(x)$ in our manifold $(M, \langle\,,\,\rangle)$ will be compared with that of a model manifolds in the sense of Greene and Wu, [9]. As we shall see, a substantial part of our investigation will be devoted to model manifolds, of which Euclidean and hyperbolic spaces are particular cases. For this reason, we feel convenient to recall their definition and basic properties.

Definition 1. *A complete Riemannian manifold $(M, \langle\,,\,\rangle)$ is called a model if there exists a point $o \in M$ such that $\exp_o : T_oM \to M$ is a diffeomorphism and the metric $\langle\,,\,\rangle$ writes, in polar geodesic coordinates (r, θ) on $M\backslash\{o\}$, as*

$$\langle\,,\,\rangle = \mathrm{d}r^2 + g(r)^2 \mathrm{d}\theta^2, \tag{14}$$

where $\mathrm{d}\theta^2$ is the standard metric on the unit sphere $\mathbb{S}^{m-1} \subseteq \mathbb{R}^m$ and $g \in C^\infty(\mathbb{R}^+_0)$ satisfies $g > 0$ on $\mathbb{R}^+$,

$$g(0) = 0, \quad g'(0) = 1, \quad g^{(2k)}(0) = 0 \text{ for } k = 1, 2, 3, \ldots, \tag{15}$$

where $g^{(j)}$ stands for the j-th iterated derivative of g.

Such a model will be denoted by M_g. If g is only required to be $C^2(\mathbb{R}^+_0)$ and g satisfies the first two requests in (15), then the model will be called a C^2-model. Note that (15) is a necessary and sufficient condition for $\langle\,,\,\rangle$ defined in (14) to extend smoothly at o. We recall that

$$K_{\rm rad} = -\frac{g''(r)}{g(r)}, \qquad v(r) \doteq \frac{\mathrm{vol}(\partial B_r)}{\omega_{m-1}} = g(r)^{m-1},$$
$$\Delta r(x) = (m-1)\frac{g'(r(x))}{g(r(x))} = \frac{v'(r(x))}{v(r(x))}, \tag{16}$$

where ω_{m-1} is the $(m-1)$-volume of the unit sphere $\mathbb{S}^{m-1}$ and ∂B_r is the boundary of the geodesic ball $B_r = \{x \in M_g \,:\, r(x) < r\}$. As the first equation in (16) suggests, one can also construct a model by specifying the radial sectional curvature $-G \in C^0(\mathbb{R}^+_0)$ and recovering g as the solution of the Cauchy problem

$$\left\{ \begin{array}{l} g'' - Gg = 0 \\ g(0) = 0, \quad g'(0) = 1. \end{array} \right. \tag{17}$$

In this case, we say that M_g is constructed from G. As important examples, the Euclidean space $\mathbb{R}^m$ can be obtained with the choices $G(r) = 0$, $g(r) = r$, while for the hyperbolic space $\mathbb{H}^m_H$ of constant sectional curvature $-H^2 < 0$ one can choose $G(r) = -H^2$, and consequently $g(r) = H^{-1}\sinh(Hr)$.

We now describe our main results. The first regards the Yamabe problem on asymptotically hyperbolic manifolds and, in some sense, can be thought as a "hyperbolic" analogue of Theorem 1. Even for $M = \mathbb{H}^m_H$, we underline that the next theorem seems to be, to the best of our knowledge, the first existence result for the Yamabe problem on $\mathbb{H}^m_H$ that allows the new scalar curvature $\widetilde{s}(x)$ to change signs. Furthermore, conditions (21) and (22) below enable $\widetilde{s}(x)$ to have very ample oscillations between positive and negative values and, as we will see below, (21) and (22) are sharp.

Theorem 2. *Let $(M, \langle\, ,\, \rangle)$ be a complete manifold of dimension $m \geq 3$, with a pole o and sectional curvature K satisfying*

$$-H^2 - \mathcal{K}\big(r(x)\big) \leq K(x) \leq -H^2, \tag{18}$$

for some non-negative $\mathcal{K} \in C^0(\mathbb{R}^+_0)$ with the property that

$$\mathcal{K}(r) \in L^1(+\infty). \tag{19}$$

Suppose that the scalar curvature $s(x)$ of M is such that

$$s(x) \geq -\frac{(m-1)^3H^2}{m-2} \qquad \text{on } M. \tag{20}$$

Then, for each smooth function $\widetilde{s}(x) \in C^\infty(M)$ satisfying

$$|\widetilde{s}(x)| \leq B(r(x)), \tag{21}$$

for some $B(r) \geq 0$ for which

$$e^{-2Hr}B(r) \in L^1(+\infty), \tag{22}$$

the metric $\langle\, ,\, \rangle$ can be conformally deformed to a smooth metric $\widetilde{\langle\, ,\, \rangle}$ of scalar curvature $\widetilde{s}(x)$, satisfying

$$\Gamma_1 e^{-2Hr(x)}\langle\, ,\, \rangle_x \leq \widetilde{\langle\, ,\, \rangle}_x \leq \Gamma_2 e^{-2Hr(x)}\langle\, ,\, \rangle_x \qquad \forall\, x \in M, \tag{23}$$

for some $0 < \Gamma_1 \leq \Gamma_2$. Furthermore, Γ_2 and consequently Γ_1 can be chosen to be as small as we wish.

Remark 4. The hyperbolic space $\mathbb{H}^m_H$ satisfies all the requirements in the above theorem. Indeed, it is enough to choose $\mathcal{K}(r) \equiv 0$ and to observe that the scalar curvature of $\mathbb{H}^m_H$ is $-m(m-1)H^2$. As Theorem 11 below will show, in this case we also have the asymptotic relation

$$\widetilde{\langle\, ,\, \rangle}_x \sim Ce^{-2Hr(x)}\langle\, ,\, \rangle_x \qquad \text{as } r(x) \to +\infty, \tag{24}$$

for some $C \in [\Gamma_1, \Gamma_2]$.

Remark 5. In view of the gap theorems in [10], one might ask if the couple of conditions (18) and (19) automatically imply that M is isometric to the hyperbolic space. The problem has been considered in a recent paper of H. Seshadri, [34]. His main result states that, if (18) holds and $e^{2Hr}\mathcal{K}(r) \to 0$ as $r \to +\infty$, then M is isometric to $\mathbb{H}^m_H$. In fact, following the proof of Theorem A in [34], one can easily weaken the condition to

$$\liminf_{r\to+\infty} e^{2Hr}\mathcal{K}(r) = 0. \tag{25}$$

The clean method in [34] may suggest that (25) is sharp, though we do not know examples to support this assertion. However, condition (19) is skew with (25).

The requirement (22) is sharp. Indeed, regarding the case $\widetilde{s}(x) \leq 0$ on $\mathbb{H}^m_H$, a considerable amount of work has been done in [31] both for existence and for non-existence of solutions of (1). The condition on $\widetilde{s}(x)$ in [31] yielding non-existence (condition (1.2), Case II in [31], see also Remark 1.1 therein) has subsequently been refined in Corollary B of [2]. This latter result states that no solutions of the Yamabe problem on $\mathbb{H}^m_H$ exist whenever

$$\widetilde{s}(x) \leq 0 \quad \text{on } \mathbb{H}^m_H, \qquad \widetilde{s}(x) \leq -C\frac{e^{2Hr(x)}}{r(x)\log r(x)} \qquad \text{for } \ r(x) >> 1 \tag{26}$$

and some constant $C > 0$, and shows the sharpness of (22). In Corollary 6, at the end of this paper, we will determine a mildly more demanding non-existence condition with a technique different from that of [2]. In the spirit of Naito's improvement of Ni's result, we feel motivated to ask the following:

> **question:** suppose that
>
> $$\widetilde{s}(x) \leq -B(r(x)) \leq 0 \qquad \text{and} \qquad e^{-2Hr}B(r) \notin L^1(+\infty),$$
>
> is it true that the Poincaré metric of $\mathbb{H}^m_H$ cannot be conformally deformed to a new metric of scalar curvature $\widetilde{s}(x)$?

Unlike (8) for Euclidean space, property (23) implies that the new metric is far from being geodesically complete. This is a consequence of the presence of the linear term $s(x)$ in (3). It would be desirable to find conformal deformations of $\langle\, ,\, \rangle$ such that the new metric gives rise to a complete manifold. For $\widetilde{s}(x) \leq 0$, the problem has been investigated in [31]. Their main result is that there exists a conformal deformation of the Poincaré metric, having scalar curvature $\widetilde{s}(x)$ and giving rise to a complete metric, provided that

$$-Cr(x)^2 \leq \widetilde{s}(x) < 0 \qquad \text{outside some ball}$$

(see Case I, Theorem 1.1 and Remark 1.1 in [31]). The next natural question arises:

question: if

$$|\widetilde{s}(x)| \le Cr(x)^2 \qquad \text{outside some ball,}$$

can the Poincaré metric of $\mathbb{H}^m_H$ be conformally deformed to a complete metric of scalar curvature $\widetilde{s}(x)$?

Notational conventions. For a function B on $\mathbb{R}$, we define B_+ and B_-, respectively, to be the positive and negative parts of B. Having fixed the reference origin $o \in M$, we denote with $r(x) = \mathrm{dist}(x, o)$. Moreover, we agree on writing $w(x) \in \mathrm{Höl}^j_{\mathrm{loc}}(M)$ if, for every relatively compact open set $\Omega \Subset M$, there exists $\alpha = \alpha(\Omega) \in (0, 1]$ such that $w \in C^{j,\alpha}(\overline{\Omega})$. For $j = 0$, we simply write $w(x) \in \mathrm{Höl}_{\mathrm{loc}}(M)$.

Theorem 2 is one of the various applications of two general existence results for Yamabe type equations, Theorems 9 and 10 below. In order to put them into the right perspective, we introduce some terminology. As a general observation, the analysis of (1) heavily depends on the spectral properties of the linear Schrödinger operator

$$L = -\Delta - q(x),$$

in particular on the sign of the bottom of its spectrum $\lambda_1^L(M)$. If this latter is negative, then the situation is more "rigid". In fact, on any Riemannian manifold,

- if $\lambda_1^L(M) < 0$ and $b(x) \le 0$, then there are no positive solutions of (1). This follows from a simple spectral argument. Indeed, by contradiction, if $u > 0$ solves (1), then u would be a positive solution of $Lu \ge 0$. By a result in [7] and [22], this is equivalent to $\lambda_1^L(M) \ge 0$, against our assumption;
- if $\lambda_1^L(M) < 0$, $b(x) \ge 0$ and the zero set of b is small in a spectral sense, then there always exists a minimal and a maximal positive solutions of (1). See for instance [28] and Section 2.4 in [1].

The case $\lambda_1^L(M) \ge 0$ seems to be the most challenging, and we will focus on it in the sequel. Sufficient conditions on $q(x)$, related to the geometry of M, yielding $\lambda_1^L(M) \ge 0$ have been obtained in [1]. In particular, our setting will be that of a complete manifold $(M, \langle\, ,\, \rangle)$ possessing a pole o and with radial sectional curvature satisfying

$$K_{\mathrm{rad}}(x) \le -G\big(r(x)\big), \tag{27}$$

for some continuous G on $\mathbb{R}^+$. We construct a model M_g based on G, and we suppose that the solution g of (17) is positive on $\mathbb{R}^+$ and satisfies the assumption

$$\frac{1}{v} \in L^1(+\infty), \qquad \text{where } v(r) = g(r)^{m-1}. \tag{V_{L1}}$$

This is equivalent to assume that M_g is a non-parabolic manifold (see [11] for details). Note that G is not required to have a sign. Indeed, some reasonable negativity of G is allowed, as the examples in (38) below show. Under condition (V_{L1}), the critical curve

$$\chi(r) = \left(2v(r)\int_r^{+\infty} \frac{\mathrm{d}s}{v(s)}\right)^{-2} = \left[\left(-\frac{1}{2}\log \int_r^{+\infty} \frac{\mathrm{d}s}{v(s)}\right)'\right]^2 \qquad \text{on } \mathbb{R}^+$$

is well defined. Hereafter, when we will need to emphasize the dependence of χ on the volume v, we shall write χ_v. In many instances, including those leading to Euclidean and hyperbolic spaces, the integral in the definition of χ can be explicitly computed, and a closed expression for χ can be found, see the next section and the Appendix for further details. For instance, if M_g is the $m \geq 3$ dimensional Euclidean space,

$$\chi(r) = \frac{(m-2)^2}{4r^2} \tag{28}$$

is the weight of the standard Hardy inequality, also called the Uncertainty Principle lemma. By Theorem 5.17 in [1], $\lambda_1^L(M) \geq 0$ whenever

$$q(x) \leq \chi(r(x)), \tag{29}$$

as a consequence of the next non-Euclidean Hardy inequality:

$$\int_M (\chi \circ r)\phi^2 \leq \int_M |\nabla \phi|^2 \qquad \text{for every } \phi \in \mathrm{Lip}_c(M).$$

The upper bound (29) will always be assumed throughout the paper.

It is worth to stress here that the expression of χ can be written as

$$\chi(r) = \frac{|\nabla \log \mathcal{G}|^2}{4},$$

where

$$\mathcal{G}(r) = \int_r^{+\infty} \frac{\mathrm{d}s}{g(s)^{m-1}}$$

is, up to a constant, the minimal positive Green kernel of the model M_g when the second point is the fixed origin o. There is a deep link between Green kernels and the techniques described in this paper and in our previous work [1]. However, to keep the length of the paper reasonably contained, we do not go into this fascinating subject any further, and we postpone its analysis to future work.

We are ready to state our first main result.

Theorem 3. *Let $(M, \langle\, ,\, \rangle)$ be a complete, m-dimensional Riemannian manifold with a pole o and radial sectional curvature K_{rad} with respect to o satisfying*

$$K_{\mathrm{rad}}(x) \leq -G\big(r(x)\big), \tag{30}$$

for some $G \in C^0(\mathbb{R}_0^+)$. Let $g \in C^2(\mathbb{R}_0^+)$ be a solution of

$$\begin{cases} g'' - Gg \leq 0 & \text{on } \mathbb{R}^+, \\ g(0) = 0, & g'(0) = 1. \end{cases} \tag{31}$$

Suppose that $g > 0$ on $\mathbb{R}^+$ and that $v = g^{m-1}$ satisfies (V_{L1}), and set $\chi = \chi_v$ as usual. Let $q(x), b(x) \in \mathrm{Höl}_{\mathrm{loc}}(M)$ be such that

$$|q(x)| \leq A(r(x)), \qquad |b(x)| \leq B(r(x)), \tag{32}$$

for some non-negative $A, B \in L^\infty_{\mathrm{loc}}(\mathbb{R}_0^+)$ with

$$A(r) \leq \chi(r) \quad \text{on } \mathbb{R}^+, \qquad \frac{A(r)}{\sqrt{\chi(r)}} \in L^1(+\infty), \qquad \frac{B(r)}{\sqrt{\chi(r)}} \in L^1(+\infty). \tag{33}$$

Fix $\sigma > 1$. Then, there exists a constant $\beta > 0$, depending on σ, g, A, B such that, for each $\gamma_\infty \in (0, \beta)$, there exist $0 < \Gamma_1 \le \Gamma_2$ and a solution $u \in \mathrm{H\ddot{o}l}^2_{\mathrm{loc}}(M)$ of

$$\Delta u + q(x)u - b(x)u^\sigma = 0 \tag{34}$$

satisfying

$$\Gamma_1 \le u(x) \le \Gamma_2 \qquad \text{on } M \tag{35}$$

and

$$\lim_{r(x)\to+\infty} u(x) = \gamma_\infty. \tag{36}$$

Moreover, $\Gamma_2 \to 0$ as $\gamma_\infty \to 0$.

We spend few words to comment on Theorem 3.

Remark 6. The choice $G(r) = 0$ allows us to include each Cartán-Hadamard manifold of dimension $m \ge 3$, and in particular Euclidean space. By the simple expression (28) for $\chi(r)$, conditions (32) and (33) read

$$\begin{array}{ll} |q(x)| \le A(r(x)) \le \dfrac{(m-2)^2}{4r(x)^2}, & rA(r) \in L^1(+\infty); \\ |b(x)| \le B(r(x)), & rB(r) \in L^1(+\infty). \end{array} \tag{37}$$

Thus, Theorem 3 improves on Ni-Naito-Kawano result even in the very special case of Euclidean space, as it enables $q(x)$ to be nonzero. On the other hand, also the case $q(x) = 0$ seems to be worth of interest for a general G, see Corollary 2 and the subsequent discussion.

Remark 7. Theorem 3 has a particular feature: as can be seen in (33), the value of the exponent $\sigma > 1$ in the nonlinearity plays no role in conditioning the growth of A and B. This is tightly related to the integrability requirement on A. Indeed, in Corollary 1 below (see also the general Theorem 10) we will see that, when $A/\sqrt{\chi}$ is non-integrable, the role of σ in the growth condition for B will be essential.

As a matter of fact, we can find solutions g of (31) for two interesting classes of functions $G(r)$:

$$\begin{array}{lll} (i) & G(r) = H^2(1+r^2)^{\alpha/2} & \text{for} \quad H > 0, \ \alpha \ge -2, \\ (ii) & G(r) = -\dfrac{H^2}{(1+r)^2} & \text{for} \quad H \in [0, 1/2]\,. \end{array} \tag{38}$$

Note that Euclidean space is in the second class by choosing $H = 0$, while the hyperbolic space $\mathbb{H}^m_H$ is in the first class by choosing $\alpha = 0$. Model manifolds constructed from a function G in the first class are negatively curved in the radial direction, while those in the second class are positively curved. For most of these G, we can find a closed, simple expression for the critical curve via explicit integration. We collect relevant computations in the Appendix.

Specializing Theorem 3 to the hyperbolic space, the second condition in (33) is equivalent to $A(r) \in L^1(+\infty)$. Unfortunately, in the geometric case of the Yamabe problem on $\mathbb{H}^m_H$ the coefficient of the linear part is

$$q(x) = -\frac{s(x)}{c_m} = \frac{m(m-2)H^2}{4},$$

thus $|q(x)| \leq A(r(x))$ and $A(r) \in L^1(+\infty)$ cannot be satisfied at the same time. This calls for a generalization of the previous result, which will be accomplished in Theorem 10 below. There, we deal with the case when $q(x)$ is close, in an appropriate integrable sense, to $k\chi(r(x))$, for some $k \in (0,1)$ (see (190)). In this respect, Theorem 3 considers the case $k = 0$.

As outlined before, the strategy of proof of Theorems 3 and 10 is based on the application of the monotone iteration scheme. Sub- and supersolutions are constructed via radialization and ODE arguments of independent interest. However, while in the first case the monotonicity of sub- and supersolutions well matches with the bounds of the Laplacian of the distance function coming from (30), when $k \in (0,1)$ the construction of the subsolution requires a tricky argument for which a lower bound on the sectional curvature of the type

$$K_{\mathrm{rad}} \geq -\bar{G}(r)$$

is also needed. This is by no means a simple change of monotonicity of the subsolution that agrees with the opposite bound on the Laplacian of the distance function, but indeed requires a delicate construction based on refined asymptotic estimates for the solutions of two associated linear Cauchy problems. For more details, see the discussion before Theorem 10. We postpone the statement of the general result, and we present here its consequence for manifolds close to the Euclidean space.

Corollary 1. *Let $(M, \langle\, ,\, \rangle)$ be a complete manifold of dimension $m \geq 3$, with a pole o and radial sectional curvature satisfying*

$$-\bar{G}\big(r(x)\big) \leq K_{\mathrm{rad}}(x) \leq 0,$$

for some non-negative $\bar{G}(r)$ with

$$r\bar{G}(r) \in L^1(+\infty).$$

Let $q(x) \in \mathrm{H\ddot{o}l}_{\mathrm{loc}}(M)$ be such that

$$0 \leq A_1\big(r(x)\big) \leq q(x) \leq A_2\big(r(x)\big),$$

for some $A_j \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+_0)$, $j \in \{1,2\}$ with

$$A_2(r) \leq \frac{(m-2)^2}{4r^2} = \chi(r).$$

Suppose that there exists $k \in (0,1)$ for which

$$r\,[A_j(r) - k\chi(r)] \in L^1(+\infty) \tag{39}$$

for each j. Let $\sigma > 1$, $B \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+_0)$ and suppose that

$$B(r) r^{-\frac{m-2}{2}(1-\sqrt{1-k})(\sigma-1)+1} \in L^1(+\infty). \tag{40}$$

Then, for each $b(x) \in \mathrm{H\ddot{o}l}_{\mathrm{loc}}(M)$ satisfying

$$|b(x)| \leq B(r(x)) \qquad \text{on } M,$$

there exists a positive solution $u \in \mathrm{H\ddot{o}l}^2_{\mathrm{loc}}(M)$ of

$$\Delta u + q(x)u - b(x)u^\sigma = 0 \qquad \text{on } M \tag{41}$$

with the property that

$$\Gamma_1 r(x)^{-\frac{m-2}{2}(1-\sqrt{1-k})} \leq u(x) \leq \Gamma_2 r(x)^{-\frac{m-2}{2}(1-\sqrt{1-k})} \qquad \text{for } r(x) \geq 1, \tag{42}$$

for some $0 < \Gamma_1 \leq \Gamma_2$. Furthermore, Γ_2 can be chosen to be as small as we wish.

Remark 8. The restriction $k \in (0,1)$ in Corollary 1 above and in Theorem 10 is due to the lack of radial symmetry of the manifold. We underline that, for models, we can cover the entire range $k \in (-\infty, 1]$, see Theorem 11 below. More precisely, if M in Corollary 1 is indeed a model, we can extend the range of k in (39), (40), (42) to $(-\infty, 1)$, while, for $k = 1$, (39) and (40) are replaced, respectively, by

$$r \log r \, [A_j(r) - \chi(r)] \in L^1(+\infty)$$

and

$$B(r) r^{-\frac{m-2}{2}(\sigma-1)+1} (\log r)^\sigma \in L^1(+\infty).$$

The behaviour (42) of u, in the conclusion of the theorem, in this latter case becomes

$$\Gamma_1 r(x)^{-\frac{m-2}{2}} \log r(x) \le u \le \Gamma_2 r(x)^{-\frac{m-2}{2}} \log r(x) \qquad \text{for } r(x) \ge 2.$$

A proof of these facts can be found in Remark 30.

We conclude this Introduction by stating a corollary of our first main Theorem 3 dealing with the case $q(x) = 0$ on negatively curved manifolds. A discussion on its sharpness gives us the possibility to illustrate some subtle phenomena.

Corollary 2. *Let M be a complete manifold of dimension $m \ge 2$, with a pole and radial sectional curvature satisfying*

$$K_{\mathrm{rad}}(x) \le -H^2 \big(1 + r(x)^2\big)^{\frac{\alpha}{2}}, \qquad \text{for some } \alpha \ge -2,\ H > 0. \tag{43}$$

Let $\sigma > 1$, and let $B \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_0^+)$ be such that

$$B(r) r^{-\frac{\alpha}{2}} \in L^1(+\infty). \tag{44}$$

Suppose that $b(x) \in \mathrm{H\ddot{o}l}_{\mathrm{loc}}(M)$ satisfies $|b(x)| \le B(r(x))$ on M. Then, there exists a constant $\beta > 0$, depending on σ, α, H, B such that, for each $\gamma_\infty \in (0, \beta)$, there exist $0 < \Gamma_1 \le \Gamma_2$ and a solution $u \in \mathrm{H\ddot{o}l}^2_{\mathrm{loc}}(M)$ of

$$\Delta u = b(x) u^\sigma \tag{45}$$

satisfying

$$\Gamma_1 \le u(x) \le \Gamma_2 \qquad \text{on } M \tag{46}$$

and

$$\lim_{r(x) \to +\infty} u(x) = \gamma_\infty. \tag{47}$$

Moreover, $\Gamma_2 \to 0$ as $\gamma_\infty \to 0$.

By Corollary 2, the mild requirement (44) on B ensures the existence of positive, bounded nonzero solutions of

$$\Delta u = b(x) u^\sigma \qquad \text{on } M. \tag{48}$$

Now, suppose that $b(x) > 0$ on M. By Theorem 3.11 in [26], and the subsequent remark after Definition 3.12, the existence of bounded, non-negative and non-constant solutions of (48) is equivalent to say that the operator $b(x)^{-1}\Delta$ does not satisfy the weak maximum principle (see [26] for relevant definitions and results, and also [21], Theorem 2.12 for generalizations). On the other hand, by Proposition 3.18 in [26],

$b(x)^{-1}\Delta$ satisfies the weak maximum principle (hence no bounded, non-negative, non-constant solutions of (48) can exist) provided that the following conditions are satisfied:

$$b(x) \ge B(r(x)) > 0 \quad \text{on } M_g, \qquad B(r) = \frac{C}{r^\mu} \quad \text{for } r >> 1, \qquad \frac{r^{1-\mu}}{\log \mathrm{vol}(B_r)} \not\in L^1(+\infty), \tag{49}$$

for some $C > 0$, $\mu \in \mathbb{R}$. Suppose that, for simplicity, $M = M_g$ is a model satisfying

$$K_{\mathrm{rad}} \le 0 \quad \text{on } M_g, \qquad \text{and} \qquad K_{\mathrm{rad}} \sim -H^2 r^\alpha \quad \text{as } r \to +\infty, \tag{50}$$

for some $H > 0$, $\alpha \ge -2$. Then, by Propositions 2.27 and 2.28 in [1] and some computations,

$$\log \mathrm{vol}(B_r) \sim \begin{cases} C r^{1+\frac{\alpha}{2}} & \text{if } \alpha > -2 \\ C \log r & \text{if } \alpha = -2, \end{cases}$$

Therefore

$$\frac{r^{1-\mu}}{\log \mathrm{vol}(B_r)} \not\in L^1(+\infty) \quad \text{if and only if } \mu \le 1 - \frac{\alpha}{2}.$$

For this choice of μ,

$$B(r) r^{-\frac{\alpha}{2}} = \frac{C}{r^{\mu+\frac{\alpha}{2}}} \not\in L^1(+\infty),$$

thus (44) barely fails to be satisfied. This proves the sharpness of (44). It is important to stress that, for the existence of bounded positive solutions of (48), the particular form $f(u) = u^\sigma$ of the nonlinearity plays no role. Indeed, u^σ could be replaced by any continuous nonlinearity $f(u)$ which is positive for $u > 0$, see [21], Theorem 2.12 for a general statement, and [30].

On the contrary, the growth of the nonlinearity in a neighbourhood of $+\infty$ is extremely important when investigating the existence of possibly unbounded, positive solution of (48). In this setting, a key role is played by the Keller-Osserman condition (which we label (KO)), independently discovered in [14] and [25] for the differential inequality $\Delta u \ge f(u)$ on $\mathbb{R}^m$. In a manifold setting and for general inequalities of the form

$$Qu \ge b(x) f(u) l(|\nabla u|), \tag{51}$$

where Q is a quasilinear operator belonging to some large class, sharp generalizations of the Keller-Osserman condition have been given in [20]. As it is apparent from the results in [20], the explicit expression of (KO) only involves the functions f, l and the structure of the operator Q. However, both the geometry of the underlying manifold and the weight $b(x)$ still play a role, which reflects into some restrictions on the range of applicability of (KO). The origin of these restrictions is still somehow obscure, and we are planning to investigate it in the future. Here, we are going to use a result in [20] to prove that, under the conditions $b(x) \ge B(r(x)) > 0$ on M_g in (50), with

$$B(r) = \frac{C}{r^\mu} \quad \text{for } r >> 1, \quad \text{and} \quad \mu \le 1 - \frac{\alpha}{2}, \tag{52}$$

(so that (49) are met, and thus $b(x)^{-1}\Delta$ satisfies the weak maximum principle on M_g) there do not exist positive solutions of (48) at all, not even unbounded. Indeed, we apply Corollary A1 of [20] with the choices $\beta = \alpha$, $p = 2$, $q = 0$, $f(t) = t^\sigma$. Condition (KO) is satisfied since $\sigma > 1$, and the inequality for μ in (52) is exactly the last requirement on the parameters in Corollary A1. Therefore, a direct application of the corollary asserts that each non-negative solution u of (48) is constant, hence zero. We suggest the interested reader to look at Proposition 3 in the last part of this paper for a result strictly related to Corollary A1 in [20].

Contents

By way of summary, we point out the three main novelties of the paper.

(1) From the PDE's and ODE's point of view, the extension, from Euclidean space to models, of Ni and Naito's idea used to generate radial sub- and supersolutions of Yamabe type equations. This requires the different and new procedures described in Theorems 6 and 7 below, in order to deal both with the linear term and with the sign-changing nonlinearity. To the best of our knowledge, the two issues together have not been considered in the literature even in the particular case of $\mathbb{R}^m$.

(2) Theorems 9 and 10, which are the first existence results for Yamabe type equations with a sign-changing nonlinearity on a general non-compact ambient space. Their validity requires the only assumptions that the manifold possesses a pole and a control on the radial sectional curvature.

(3) The asymptotic estimates for the solutions of the linear ODE, contained in Theorem 5. Besides being interesting in their own, they allow us to effectively express the various assumptions of Theorems 6 and 7 in terms of the bounds on the curvature for instance expressed in (38) (see the Appendix for detailed computations).

As a matter of fact, the role of the critical curve will be central and ubiquitous in this work. In this spirit, the present paper constitutes a natural continuation of our previous [1]. There, the interested reader can find a number of other geometric problems where the study of $\chi(r)$ turns out to be extremely useful.

Estimates for the linear Cauchy problem

Let M_g be a model of dimension $m \geq 2$. Set $v = g^{m-1}$, and let $A \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+_0)$. This section is devoted to the proof of asymptotic estimates for a solution $h \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^+_0)$ of the Cauchy problem

$$\begin{cases} (vh')' + Avh = 0 & \text{on } \mathbb{R}^+ \\ h(0) = 1, \qquad h'(0) = 0. \end{cases} \tag{53}$$

The existence and uniqueness theory for solutions of (53) is quite classical, and the reader can find the basic results for the $\mathrm{Lip}_{\mathrm{loc}}$ class in Section 3 of [1].

As we shall see, sharp conditions on $b(x)$ establishing existence or non-existence for positive solutions of the Yamabe type equation (1) will be given in terms of the behaviour of positive solutions h of the linear part $\Delta h + q(x)h = 0$. This is the case, for instance, of (133) in Theorem 7, in the radial setting. The search for a precise control of the asymptotic behaviour of h is thus necessary to rewrite our conditions in a more effective, but still sharp, form. The analysis of (53) heavily depends on whether we are in a parabolic setting or not. Hereafter, we suppose that $v(r)$ satisfies the following assumption:

$$\frac{1}{v} \in L^1(+\infty). \tag{V_{L1}}$$

As stressed in the Introduction, this is equivalent to require that M_g be non-parabolic (see [11] for instance). The study of (53) in the non-parabolic setting turns out to be more delicate and rich of subtleties than in the parabolic case. In [1] we have shown the key role played, in this analysis, by the critical curve of v, $\chi = \chi_v$, defined as

$$\chi(r) = \left(2v(r)\int_r^{+\infty}\frac{\mathrm{d}s}{v(s)}\right)^{-2} = \left[\left(-\frac{1}{2}\log\int_r^{+\infty}\frac{\mathrm{d}s}{v(s)}\right)'\right]^2 \qquad \text{on } \mathbb{R}^+.$$

Note that, by a first integration,

$$\sqrt{\chi(r)} \notin L^1(+\infty) \tag{54}$$

Example 1. In the particular cases of Euclidean and hyperbolic space, χ has the following expressions:

- For $\mathbb{R}^m$, where $g(r) = r$, (V_{L1}) is satisfied if and only if $m \geq 3$, and

$$\chi(r) = \frac{(m-2)^2}{4r^2}.$$

- In the hyperbolic space $\mathbb{H}^m_H$, where $g(r) = H^{-1}\sinh(Hr)$, denote with χ_m the critical curve for $\mathbb{H}^m_H$. A recursive formula, which can be easily proved integrating by parts the definition of χ_m, enables us to compute χ_m from χ_2 and χ_3; indeed:

$$\frac{m-1}{2\sqrt{\chi_m(r)}} = \frac{\coth(Hr)}{H} - \frac{1}{\sinh^2(Hr)}\frac{m}{2\sqrt{\chi_{m+2}(r)}}.$$

By explicit integration,

$$\begin{aligned} \chi_2(r) &= H^2\left[2\sinh(Hr)\log\left(\frac{e^{Hr}+1}{e^{Hr}-1}\right)\right]^{-2}; \\ \chi_3(r) &= \frac{H^2}{(1-e^{-2Hr})^2}. \end{aligned} \tag{55}$$

Details can be found in [1], Example 3.15. For future use, we remark that the following properties hold (see [1], Remark 3.19 and Proposition 3.23):

$$\chi_m(r) > \frac{H^2(m-1)^2}{4} \quad \text{on } \mathbb{R}^+, \qquad \chi_m(r) \to \frac{H^2(m-1)^2}{4} \quad \text{as } r \to +\infty. \tag{56}$$

Moreover, a tedious but straightforward computation shows that

$$\chi(r) = \frac{1}{4}\left\{\frac{1}{(m-1)H} + \frac{m-1}{(m+1)H}e^{-2Hr} + o\left(e^{-2Hr}\right)\right\}^{-2} \tag{57}$$

For notational convenience, for $k \in (-\infty, 1]$, we define the function $H_k(r)$ as follows:

$$
\begin{aligned}
H_1(r) &= -\sqrt{\int_r^{+\infty} \frac{\mathrm{d}s}{v(s)} \log \int_r^{+\infty} \frac{\mathrm{d}s}{v(s)}} \qquad && k = 1 \\
H_k(r) &= \left[\int_r^{+\infty} \frac{\mathrm{d}s}{v(s)}\right]^{(1-\sqrt{1-k})/2} && k \in (-\infty, 1)
\end{aligned} \tag{58}
$$

Note that the first one is positive only for sufficiently large r.

Remark 9. We observe that, for each $k \le 1$, $(H_k^2 v)^{-1} \in L^1(+\infty)$. Clearly, it is enough to prove the result for $k = 1$, and the assertion can be checked by changing variables in the integral

$$
\int^{+\infty} \frac{\mathrm{d}t}{H_1(t)^2 v(t)} = -\int^{+\infty} \frac{1}{v(t)} \left(\int_t^{+\infty} \frac{\mathrm{d}s}{v(s)}\right)^{-1} \log^{-2}\left(\int_t^{+\infty} \frac{\mathrm{d}s}{v(s)}\right) \mathrm{d}t
$$

according to

$$
x(t) = \left(\int_t^{+\infty} \frac{\mathrm{d}s}{v(s)}\right)^{-1}.
$$

The above remark enables us to construct the critical curve with respect to the weighted volume $H_k^2 v$,

$$
\chi_{H_k^2 v}(r) = \left(2H_k^2(r) v(r) \int_r^{+\infty} \frac{\mathrm{d}s}{H_k^2(s) v(s)}\right)^{-2}.
$$

With a simple computation, one can check that

$$
\begin{cases}
\chi_{H_k^2 v}(r) = (1-k)\chi(r) & \text{if } k < 1, \\
\chi_{H_1^2 v}(r) = \chi(r) \left(\log \displaystyle\int_r^{+\infty} \frac{\mathrm{d}s}{v(s)}\right)^{-2} & \text{for } k = 1.
\end{cases} \tag{59}
$$

Our starting point is the following result (see Theorem 5.2 and Proposition 5.7 of [1])

Theorem 4. *Suppose that $A(r) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_0^+)$, and let $v(r)$ be as above and satisfying (V_{L1}). Let χ be the critical curve of v and assume that*

$$
A(r) \le k\chi(r) \qquad \text{on } \mathbb{R}_0^+, \ \text{ for some } \ k \in (-\infty, 1]. \tag{60}
$$

Then, the solution $h(r) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_0^+)$ of

$$
\begin{cases}
(vh')' + Avh = 0 \qquad \text{on } \mathbb{R}^+ \\
h(0) = 1, \qquad h'(0) = 0
\end{cases} \tag{61}
$$

is positive on $\mathbb{R}_0^+$ and there exist $r_1 > 0$ sufficiently large and a constant $C = C(r_1) > 0$ such that

$$
h(r) \ge CH_k(r) \tag{62}
$$

on $[r_1, +\infty)$. If $A \equiv k\chi$ for some $[r_2, +\infty)$, then we can replace (62) with

$$
h(r) \sim CH_k(r) \qquad \text{as } r \to +\infty \tag{63}
$$

for some constant $C > 0$.

Remark 10. Before the recent [1], in the literature result of this type that we are aware of have only been obtained in [3] (Lemma 2.3 and Remark 2.4), for Euclidean type $v(r)$, and in [2] (Theorem 3.2) in a hyperbolic setting.

Although we shall not presently prove the above theorem in its full strength, we nevertheless give a sketch of a geometrical proof of the fact that $h > 0$ on $\mathbb{R}^+$ via an argument which relies on an observation of P. Li and J. Wang [17]. Consider on M_g, with coordinates (r, θ), $\theta \in \mathbb{S}^{m-1}$, the minimal positive Green function evaluated on the pair $(o, (r, \theta))$:

$$\mathcal{G}\big((r,\theta)\big) = \mathcal{G}(r) = \int_r^{+\infty} \frac{\mathrm{d}s}{v(s)}$$

(up to multiplication by an unessential constant). Then, $\mathcal{G}$ is positive, harmonic on $M_g \backslash \{o\}$ and with a singularity at o. Note that $\mathcal{G}$ exists by the non-parabolicity condition (V_{L1}). Then, for every $a \in \mathbb{R}^+$ the function $\mathcal{G}_a = \min\{\mathcal{G}, a\}$ is positive, bounded on M_g and it is a weak solution of $\Delta \mathcal{G}_a \leq 0$. A computation shows that $f = \sqrt{\mathcal{G}_a}$ is a positive, weak solution of

$$\Delta f + \frac{|\nabla \log \mathcal{G}_a|^2}{4} f \leq 0.$$

By a result in [7] and [22], for every $\phi \in \mathrm{Lip}_c(M)$

$$\int \frac{|\nabla \log \mathcal{G}_a|^2}{4} \phi^2 \leq \int |\nabla \phi|^2, \tag{64}$$

and letting $a \to +\infty$, by monotone convergence we get

$$\int \frac{|\nabla \log \mathcal{G}|^2}{4} \phi^2 \leq \int |\nabla \phi|^2 \qquad \forall\, \phi \in \mathrm{Lip}_c(M_g). \tag{65}$$

It is immediate to verify that

$$\frac{|\nabla \log \mathcal{G}|^2}{4} = \chi(r).$$

By Rayleigh characterization it follows that, for any $A(r) \leq \chi(r)$, inequality (65) implies

$$\lambda_1^L(M_g) \geq 0, \qquad \text{where} \quad L = \Delta + A(r). \tag{66}$$

If a solution $h(r)$ of (61) has a first zero, say at some $R > 0$, then the function $u\big((r,\theta)\big) = h(r)$ would be a solution of

$$\begin{cases} \Delta u + A(r) u = 0 & \text{on } B_R \\ u = 0 & \text{on } \partial B_R, \end{cases}$$

with $u > 0$ on B_R, proving that $\lambda_1^L(B_R) \leq 0$. By the strict monotonicity property of eigenvalues, $\lambda_1^L(M_g) < 0$, contradicting (66).

Estimates (62) and (63) are sharp. However, for (63) to hold, the requirement $A = k\chi$ after some r_2 is definitely too demanding, thus it would be desirable to relax the assumption. However, simple examples show that $A \sim k\chi$ is far from being enough to guarantee (63). We report here one of them

Example 2. On Euclidean space $\mathbb{R}^m$, choose $A(r)$ satisfying $A(r) \geq \chi(r)$ on $[1, +\infty)$ and

$$A(r) = \left(\sqrt{\chi(r)} + \frac{1}{r \log r} \right)^2 \qquad \text{on } [2, +\infty).$$

Then, $\sqrt{A} - \sqrt{\chi} \notin L^1(+\infty)$, thus by Theorem 5.5 in [1] h oscillates, that is, it has infinitely many zeroes. On the other hand, $A \sim \chi$ as $r \to +\infty$.

We begin with a simple observation that will be repeatedly used throughout the paper.

Remark 11. By Remark 9, for each $k \leq 1$ it holds

$$\frac{1}{H_k^2 v} \in L^1(+\infty). \tag{67}$$

In particular, if h solves

$$\begin{cases} (vh')' + Avh = 0 & \text{on } \mathbb{R}^+ \\ h(0) = 1, \qquad h'(0) = 0, \end{cases}$$

and $A \leq \chi$, then by Theorem 4 $h > 0$ on $\mathbb{R}_0^+$ and $h(r) \geq CH_1(r)$ on $[r_1, +\infty)$, for some $r_1 >> 1$, and thus

$$\frac{1}{h^2 v} \leq \frac{C^{-2}}{H_1^2 v} \in L^1(+\infty).$$

The next Proposition allows to compare the asymptotic behaviour of the solution h of (53) with that of the solution $\hat{h}$ of another Cauchy problem, obtained perturbing the potential A to a new potential $\hat{A}$.

Proposition 1. *Let $v = g^{m-1}$ and suppose that (V_{L1}) is met. Let $h, \hat{h}$ be solutions, respectively, of*

$$\begin{cases} (vh')' + Avh = 0 \\ h(0) = 1, \quad h'(0) = 0 \end{cases}, \qquad \begin{cases} (v\hat{h}')' + \hat{A}v\hat{h} = 0 \\ \hat{h}(0) = 1, \quad \hat{h}'(0) = 0. \end{cases}$$

Assume that $A \leq \chi$ on $\mathbb{R}^+$. Having defined

$$\Lambda_{\pm} = \frac{1}{2} \int_0^{+\infty} \frac{\big(A(s) - \hat{A}(s)\big)_{\pm}}{\sqrt{\chi_{h^2 v}(s)}} \mathrm{d}s, \tag{68}$$

suppose that $\Lambda_+ < 1$ and $\Lambda_- < 1$. Then, $h, \hat{h}$ are positive on $\mathbb{R}_0^+$ and there exists $C \in [1 - \Lambda_-, (1 - \Lambda_+)^{-1}]$ such that

$$\hat{h}(r) \sim Ch(r) \qquad \text{as } r \to +\infty. \tag{69}$$

Proof. Since $A \leq \chi$, then $h > 0$ by Theorem 4. Define

$$\tilde{v}(r) = h(r)^2 v(r),$$

and note that, by Remark 11, $A \leq \chi$ implies that $1/\tilde{v} \in L^1(+\infty)$, thus $\chi_{\tilde{v}} = \chi_{h^2 v}$ is well defined. Setting $B(r) = A(r) - \hat{A}(r)$, $\hat{h}$ solves $(v\hat{h}')' + Av\hat{h} = Bv\hat{h}$. Then, $\xi = \hat{h}/h$ solves

$$\begin{cases} (h^2 v \xi')' = B h^2 v \xi & \text{on } \mathbb{R}^+ \\ \xi(0) = 1, \quad \xi'(0) = 0. \end{cases} \tag{70}$$

By definition,
$$\Lambda_\pm = \frac{1}{2}\int_0^{+\infty} \frac{B_\pm(r)}{\sqrt{\chi_{\tilde v}(r)}}\mathrm{d}r. \tag{71}$$
The idea is to construct a sub and a supersolution for (70) that squeeze at infinity to a constant C. To do so, we operate the change of variables
$$s(r) = \left(\int_r^{+\infty} \frac{\mathrm{d}t}{\tilde v(t)}\right)^{-1}, \tag{72}$$
$s : \mathbb{R}^+ \to \mathbb{R}^+$, $s' > 0$. Denoting with $r(s)$ its inverse,
$$\dot r(s) = \tilde v(r(s))\left(\int_{r(s)}^{+\infty} \frac{\mathrm{d}t}{\tilde v(t)}\right)^2 = \frac{\big[\chi_{\tilde v}(r(s))\big]^{-1/2}}{2s}, \tag{73}$$
where $\cdot$ means the derivative with respect to s. It can be checked that ξ is a solution of (70) if and only if $y(s) = s\xi(r(s))$ solves
$$\begin{cases} \ddot y(s) = \dfrac{\tilde v^2(r(s))}{s^3}B(r(s))\dfrac{y(s)}{s} & \text{on } \mathbb{R}^+. \\ y(0) = 0, \quad \dot y(0) = 1. \end{cases} \tag{74}$$
Define γ_m, γ_M according to
$$\gamma_M = \frac{1}{1-\Lambda_-} \in [1,+\infty), \qquad \gamma_m = 1-\Lambda_+ \in (0,1]. \tag{75}$$
Next, we consider the problem
$$\begin{cases} \ddot y(s) = -\dfrac{\tilde v^2(r(s))}{s^3}B_-(r(s))\gamma_M & \text{on } \mathbb{R}^+ \\ y(0) = 0, \quad \dot y(0) = \gamma_M. \end{cases} \tag{76}$$
Thus, integrating once, changing variables using (73) and recalling (72) we obtain
$$\dot y(s) = \gamma_M - \gamma_M\int_0^{r(s)} \tilde v(\rho)\frac{B_-(\rho)}{s(\rho)}\mathrm{d}\rho = \gamma_M\left[1 - \frac{1}{2}\int_0^{r(s)}\frac{B_-(\rho)}{\sqrt{\chi_{\tilde v}(\rho)}}\mathrm{d}\rho\right]. \tag{77}$$
Therefore,
$$\dot y(s) \ge \gamma_M(1-\Lambda_-) = 1$$
and, since $r(s) \to +\infty$ as $s \to +\infty$, $\dot y(s) \downarrow 1$ as $s \to +\infty$. The function y is thus increasing and, since $y(0) = 0$, y is positive on $\mathbb{R}^+$. From (77), integrating we also deduce
$$s \le y(s) \le \gamma_M s \qquad \text{on } \mathbb{R}^+ \tag{78}$$
and, from De l'Hopital's rule, $y(s) \sim s$ as $s \to +\infty$. Setting
$$\xi_+(r) = \frac{y(s(r))}{s(r)}, \tag{79}$$
ξ_+ is positive on $\mathbb{R}^+$, and
$$\xi_+'(r) = \frac{s'(r)}{s^2(r)}\Big\{\dot y(s(r))s(r) - y(s(r))\Big\} = \frac{s'(r)}{s^2(r)}\mu(s(r)). \tag{80}$$

Studying the function $\mu(s) = s\dot{y}(s) - y(s)$ we have $\mu(0) = 0$ and $\dot{\mu}(s) = s\ddot{y}(s) \leq 0$ on $\mathbb{R}^+$. It follows that $\mu \leq 0$ and so ξ_+ is non-increasing on $\mathbb{R}^+$. Furthermore, $\xi_+(0^+) = \gamma_M$ and $\xi_+(r) \downarrow 1$ as $r \to +\infty$. From the change of variables (72) and the fact that ξ_+ is positive it also follows that ξ_+ satisfies

$$\begin{cases} (\tilde{v}\xi_+')' = -B_-\tilde{v}\gamma_M \leq -B_-\tilde{v}\xi_+ \leq B\tilde{v}\xi_+ & \text{on } \mathbb{R}^+ \\ \xi_+(0) = \gamma_M, \quad \xi_+'(0) = 0. \end{cases} \tag{81}$$

Analogously, consider the problem

$$\begin{cases} \ddot{y}(s) = \dfrac{\tilde{v}^2(r(s))}{s^3} B_+(r(s)) & \text{on } \mathbb{R}^+ \\ y(0) = 0, \quad \dot{y}(0) = \gamma_m. \end{cases} \tag{82}$$

Integrating the equation on $[0, s]$ and using (75) we get

$$\dot{y}(s) = \gamma_m + \frac{1}{2}\int_0^{r(s)} \frac{B_+(\rho)}{\sqrt{\chi_{\tilde{v}}(\rho)}} \mathrm{d}\rho \leq \gamma_m + \Lambda_+ = 1,$$

and $\dot{y}(s) \to 1$ as $s \to +\infty$, from which we deduce $y(s) \sim s$ as $s \to +\infty$ via De l'Hopital theorem. From $y(0) = 0$, $\dot{y}(0) = \gamma_m$ and since $\dot{y}$ is non-decreasing being $\ddot{y} \geq 0$, we argue that

$$\gamma_m s \leq y(s) \leq s.$$

We define $\xi_-(r) = y(s(r))/s(r)$. Then, proceeding as in the case of ξ_+ , ξ_- turns out to be positive, non-decreasing, $\xi_-(0) = \gamma_m$ and $\xi_-(r) \uparrow 1$ as $r \to +\infty$. From our change of variables, ξ_- satisfies

$$\begin{cases} (\tilde{v}\xi_-')' = B_+\tilde{v} \geq B_+\tilde{v}\xi_- \geq B\tilde{v}\xi_- & \text{on } \mathbb{R}^+ \\ \xi_-(0) = \gamma_m, \quad \xi_-'(0) = 0. \end{cases} \tag{83}$$

Furthermore,

$$\gamma_m \leq \xi_-(r) \leq 1 \leq \xi_+(r) \leq \gamma_M.$$

By the monotone iteration scheme, [33], there exists $\gamma_0 \in [\gamma_m, \gamma_M]$ and a solution ξ of

$$\begin{cases} (\tilde{v}\xi')' = B\tilde{v}\xi & \text{on } \mathbb{R}^+ \\ \xi(0) = \gamma_0, \quad \xi'(0) = 0 \end{cases}$$

such that

$$\xi_- \leq \xi \leq \xi_+.$$

In particular, $\xi > 0$ and $\xi(r) \to 1$ as $r \to +\infty$. By uniqueness of solutions of linear ODE, $\xi(r)/\gamma_0$ coincides with the solution $\hat{h}/h$ of (70). Thus

$$\lim_{r \to +\infty} \frac{\hat{h}(r)}{h(r)} = \frac{1}{\gamma_0} \in \left[\gamma_M^{-1}, \gamma_m^{-1}\right] = \left[1 - \Lambda_-, \frac{1}{1 - \Lambda_+}\right],$$

and (69) is met. Furthermore, since $\hat{h} = h\xi/\gamma_0$, $\hat{h} > 0$ on $\mathbb{R}^+$, completing the proof. $\square$

Remark 12. We spend a few words to comment on the relations $\Lambda_+ < 1$ and $\Lambda_- < 1$. Indeed, there is another way to prove that these inequalities imply $\hat{h} > 0$ on $\mathbb{R}^+$. Let z be a solution of $(\bar{v}z')' + \bar{A}\bar{v}z = 0$, $z(0) = 1$, $z'(0) = 0$. According to [1], Corollary 5.42, the number $n(z)$ of zeroes of z is bounded above as follows:

$$n(z) \le \inf_{p\in[1,+\infty)} \left[\left(\frac{2p-1}{2p} \right)^{2p-1} \int_0^{+\infty} \frac{\bar{A}_+(s)^p}{\chi_{\bar{v}}(s)^{p-1/2}} \mathrm{d}s \right]. \tag{84}$$

In particular, applying the corollary with the choices $\bar{v} = h^2 v$, $z = \xi = \hat{h}/h$, $\bar{A} = -B$, and choosing $p = 1$ we deduce that

$$n(\xi) \le \frac{1}{2} \int_0^{+\infty} \frac{\big(-B(s)\big)_+}{\sqrt{\chi_{h^2v}(s)}} \mathrm{d}s = \Lambda_-.$$

Hence, if $\Lambda_- < 1$ then z is positive on $\mathbb{R}^+$, so $\hat{h} = hz$ is positive. However, the asymptotic relation $\hat{h} \sim Ch$ seems hard to deduce from the method used to prove (84).

Conditions $\Lambda_+ < 1$ and $\Lambda_- < 1$, to be verified, require the knowledge of h, as well as the explicit expression of χ_{h^2v}, on the whole $\mathbb{R}^+$. In the general case such a precise information is hardly available, thus it would be highly desirable to weaken the assumption while maintaining the asymptotic conclusion (69), perhaps with a less stringent control on the constant C. This will be achieved in the next theorem, which constitutes the main result of this section.

Theorem 5. *Let $v = g^{m-1}$ and suppose that (V_{L1}) is met. Let $A \le \chi$ on $\mathbb{R}^+$, and let h be the positive solution of*

$$\begin{cases} (vh')' + Avh = 0 & \text{on } \mathbb{R}^+, \\ h(0) = 1, \quad h'(0) = 0. \end{cases}$$

If, for some $k \le 1$,

$$\frac{A(r) - k\chi(r)}{\sqrt{\chi_{H_k^2 v}(r)}} \in L^1(+\infty), \tag{85}$$

then there exists $C > 0$ such that $h(r) \sim CH_k(r)$ as $r \to +\infty$.

Proof. Note that $h > 0$ on $\mathbb{R}^+$ is a consequence of $A \le \chi$ and Theorem 4. The proof is divided in two main steps. First, we prove the result under the further assumption

$$A(r) \le k\chi(r) \qquad \text{on } \mathbb{R}^+.$$

For each $j = 1, 2, \ldots$, choose a function $A_j \in C^0(\mathbb{R}_0^+)$ such that

$$\begin{cases} A(r) \le A_j(r) \le k\chi(r) & \text{on } \mathbb{R}_0^+ \\ A_j(r) = A(r) & \text{on } (0, j] \\ A_j(r) = k\chi(r) & \text{on } [j+1, +\infty). \end{cases} \tag{86}$$

Observe that $\{A_j\}$ is a monotone decreasing sequence of functions. We let h_j be the solution of

$$\begin{cases} (vh_j')' + A_j v h_j = 0 \\ h_j(0) = 1, \quad h_j'(0) = 0. \end{cases}$$

Then, again by Theorem 4, $h_j > 0$ on $\mathbb{R}_0^+$. Moreover, by Sturm type arguments, $\{h_j\}$ is a monotone non-decreasing sequence of functions, $h_j \le h$ for every j, and

$$\frac{h_1'}{h_1} \le \frac{h_j'}{h_j} \le \frac{h'}{h}. \tag{87}$$

One can simply see this by integrating the derivative of $(vh_j'h - vh'h_j)$ and using the initial conditions. Furthermore, from the inequalities in (87) we obtain that

$$\frac{h_j}{h}, \quad \frac{h_1}{h_j}, \qquad \text{are non-increasing on } \mathbb{R}^+,$$

so that

$$\frac{h_j^2 v}{h^2 v}, \quad \frac{h_1^2 v}{h_j^2 v} \qquad \text{are non-increasing on } \mathbb{R}^+.$$

By Remark 11, the critical curves $\chi_{h_1^2 v}$, $\chi_{h_j^2 v}$ and $\chi_{h^2 v}$ are well defined and, furthermore, we can apply Proposition 4.13 of [1] to deduce that

$$\chi_{h_1^2 v}(r) \le \chi_{h_j^2 v}(r) \le \chi_{h^2 v}(r) \qquad \text{on } \mathbb{R}^+. \tag{88}$$

Note that, by (86) and Theorem 4, for each j,

$$h_j(r) \sim CH_k(r) \qquad \text{as } r \to +\infty, \tag{89}$$

for some positive constant C clearly depending on j. Our aim is to show that, for an appropriate choice of j large, the quantities

$$(\Lambda_\pm)_j = \frac{1}{2}\int_0^{+\infty} \frac{\big(A(s) - A_j(s)\big)_\pm}{\sqrt{\chi_{h^2 v}(s)}} \mathrm{d}s,$$

satisfy $(\Lambda_+)_j < 1$, $(\Lambda_-)_j < 1$. Trivially $(\Lambda_+)_j = 0$ because of our choice of A_j. We are thus left to analyze $(\Lambda_-)_j$. By (88),

$$(\Lambda_-)_j \le \frac{1}{2}\int_0^{+\infty} \frac{A_j(s) - A(s)}{\sqrt{\chi_{h_1^2 v}(s)}} \mathrm{d}s = \overline{\Lambda}_j. \tag{90}$$

We first prove that, for each j, $\overline{\Lambda}_j < +\infty$. From (89) we know that $h_1(r) \sim CH_k(r)$ as $r \to +\infty$. Whence, $\chi_{h_1^2 v} \sim \chi_{H_k^2 v}$ follows by the very definition of χ, and

$$\frac{A_j(r) - A(r)}{\sqrt{\chi_{h_1^2 v}(r)}} \sim \frac{A_j(r) - A(r)}{\sqrt{\chi_{H_k^2 v}(r)}} = \frac{k\chi(r) - A(r)}{\sqrt{\chi_{H_k^2 v}(r)}},$$

the last equality being true for $r \ge j+1$. Now, $\overline{\Lambda}_j < +\infty$ follows from assumption (85). Since the sequence of integrands in the definition of $\overline{\Lambda}_j$ is monotone non-increasing, and $\overline{\Lambda}_1 < +\infty$, we can apply Lebesgue convergence theorem to deduce that $\overline{\Lambda}_j \to 0$ as $j \to +\infty$. Fix j_0 such that $\overline{\Lambda}_{j_0} < 1$. Then, $(\Lambda_-)_{j_0} < 1$ by (90), and applying Proposition 1 with the choice $\hat{A} = A_{j_0}$ we deduce the existence of a constant $C > 0$ such that $h(r) \sim Ch_{j_0}(r)$ as $r \to +\infty$. Combining with (89) the conclusion follows.

In the general case, that is, when $A \le k\chi$ is not guaranteed, we set

$$\bar{A}(r) = \min\big\{A(r), k\chi(r)\big\},$$

and we define $\bar{h}$ to be the solution of the linear Cauchy problem with potential $\bar{A}$. By Sturm comparison, $h \le \bar{h}$ and, since $\bar{A} \le k\chi$, by the previous case

$$\bar{h}(r) \sim CH_k(r) \qquad \text{as } r \to +\infty. \tag{91}$$

We would like to apply Proposition 1 with the "reversed" choices $h = \bar{h}$ and $\hat{h} = h$. This would be directly possible if we knew that $\Lambda_\pm$ are less than 1 in (68). Since we do not, we proceed as before and we construct the decreasing sequence of functions $\{A_j\}$ satisfying

$$\begin{cases} \bar{A}(r) \le A_j(r) \le A(r) & \text{on } \mathbb{R}_0^+ \\ A_j(r) = \bar{A}(r) & \text{on } (0, j] \\ A_j(r) = A(r) & \text{on } [j+1, +\infty), \end{cases} \tag{92}$$

and the corresponding set $\{h_j\}$ of solutions of linear Cauchy problems. As before, $\{h_j\}$ is monotone non-decreasing and, by Sturm arguments and Proposition 4.13 in [1],

$$\chi_{h^2v}(r) \le \chi_{h_1^2v}(r) \le \chi_{h_j^2v}(r) \le \chi_{\bar{h}^2v}(r).$$

We consider the quantities

$$(\Lambda_\pm)_j = \frac{1}{2}\int_0^{+\infty} \frac{\big(\bar{A}(s) - A_j(s)\big)_\pm}{\sqrt{\chi_{\bar{h}^2v}(s)}}\mathrm{d}s.$$

From the definition of $A_j(r)$, $(\Lambda_+)_j = 0$ and from (92) the functions $A_j(r) - \bar{A}(r)$ are bounded above by

$$A(r) - \bar{A}(r) = \big(A(r) - k\chi(r)\big)_+.$$

By (91),

$$\frac{A_j(s) - \bar{A}(s)}{\sqrt{\chi_{\bar{h}^2v}(s)}} \le \frac{\big(A(s) - k\chi(s)\big)_+}{\sqrt{\chi_{\bar{h}^2v}(s)}} \sim \frac{\big(A(s) - k\chi(s)\big)_+}{\sqrt{\chi_{H_k^2v}(s)}}$$

as $s \to +\infty$. Using assumption (85), we can apply Lebesgue convergence theorem to deduce that $(\Lambda_-)_j \to 0^+$ as $j \to +\infty$. Choosing j_0 large enough that $(\Lambda_-)_{j_0} < 1$, and applying Proposition 1 we conclude

$$h_{j_0}(r) \sim C\bar{h}(r) \sim CH_k(r) \qquad \text{as } r \to +\infty.$$

We are left to prove that $h_{j_0} \sim Ch$. To do this, observe that the potential A_{j_0} coincides with A on $[j_0 + 1, +\infty)$. Consider the function $\xi = h_{j_0}/h$. Then,

$$(h^2v\xi')' = Bh^2v\xi, \qquad \text{where } B = A - A_{j_0} \ge 0$$

and B has compact support. A first integration using $\xi'(0) = 0$ shows that ξ is non-decreasing, and for $r \ge j_0 + 1$

$$(h^2v\xi')(r) = (h^2v\xi')(j_0 + 1),$$

thus,

$$\xi'(r) = \frac{C}{h^2(r)v(r)} \qquad \text{on } [j_0 + 1, +\infty).$$

Integrating again,

$$\xi(r) = \xi(j_0 + 1) + C\int_{j_0+1}^r \frac{\mathrm{d}s}{h(s)^2v(s)}.$$

Letting $r \to +\infty$ and using Remark 11, we conclude that ξ is bounded. Being non-decreasing, there exists $C > 0$ such that $\xi(r) \to C$ as $r \to +\infty$. Therefore,

$$h_{j_0}(r) \sim Ch(r) \qquad \text{as } r \to +\infty,$$

concluding the proof. $\square$

Remark 13. The case $k = 0$ is particularly important. By Theorem 5, h tends to a positive constant provided that

$$A(r) \le \chi(r) \quad \text{on } \mathbb{R}^+, \text{ and } \quad \frac{A(r)}{\sqrt{\chi(r)}} \in L^1(+\infty).$$

In the Euclidean case $v(r) = r^{m-1}$, $m \ge 3$, the condition reads

$$A(r) \le \frac{(m-2)^2}{4r^2} \quad \text{on } \mathbb{R}^+, \text{ and } \quad rA(r) \in L^1(+\infty).$$

We remark that, for the very special case of a Euclidean type $v(r)$, in Lemma 2.6 of [3] (specialized to the case $A \ge 0$) the weaker bound $1 \ge h \ge C > 0$ has been obtained under the stronger requirement

$$0 \le A(r) \le \min\left\{\frac{(m-2)^2}{4r^2}, \frac{\bar{C}}{r^{2+\varepsilon}}\right\},$$

for some $\bar{C} > 0$ and $\varepsilon > 0$. We stress that Theorem 5, besides being more general and suited for non-Euclidean environments, is proved with a much simpler technique than the one developed to get Lemma 2.6 in [3].

Remark 14. When $k < 1$, by (59) condition (85) is equivalent to

$$\frac{A(r) - k\chi(r)}{\sqrt{\chi(r)}} \in L^1(+\infty), \tag{93}$$

whence there is no need to actually compute the asymptotic behaviour of $\chi_{H_k^2 v}(r)$. Note that A cannot simultaneously satisfy (93) for two different values of $k < 1$, say, k_1 and k_2. Indeed, otherwise, subtracting

$$\frac{A(r) - k_2\chi(r)}{\sqrt{\chi(r)}} \qquad \text{from} \qquad \frac{A(r) - k_1\chi(r)}{\sqrt{\chi(r)}},$$

we would have $(k_1 - k_2)\sqrt{\chi(r)} \in L^1(+\infty)$, contradicting (54). With a little more effort, it is not hard to see that condition (85) for $k = 1$ is not contained in any of the ones with $k < 1$.

To better appreciate the above result, we specialize it to the Euclidean setting:

Corollary 3. *Let $v = r^{m-1}$, $m \ge 3$. Let $A \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+_0)$ satisfying*

$$A(r) \le \frac{(m-2)^2}{4r^2} = \chi(r) \qquad \text{on } \mathbb{R}^+.$$

and either

$$i) \qquad r \log r \,[A(r) - \chi(r)] \in L^1(+\infty)$$

or, for some $k \in (-\infty, 1)$,

$$ii) \qquad r\,[A(r) - k\chi(r)] \in L^1(+\infty).$$

Let h *be a solution of*

$$\begin{cases} (vh')' + Avh = 0 & \text{on } \mathbb{R}^+, \\ h(0) = 1, \qquad h'(0) = 0. \end{cases}$$

Then, there exists $C > 0$ *such that*

$$h(r) \sim Cr^{-\frac{m-2}{2}} \log r \qquad \text{in case } i);$$

$$h(r) \sim Cr^{-\frac{m-2}{2}(1-\sqrt{1-k})} \qquad \text{in case } ii)$$

as $r \to +\infty$.

Proof. It follows from Theorem 5 by an explicit computations of $H_k(r)$ and $\chi_{H_k^2 v}(r)$. These can be found in the Appendix, Class (ii), in the case $H = 0$. □

We now state the next corollary of Theorem 5, which will be one of the cornerstones in the proof of our existence Theorem 10.

Corollary 4. *Let* $G, \bar{G} \in C^0(\mathbb{R}_0^+)$*, and let* $g, \bar{g}$ *be solutions of*

$$\begin{cases} g'' - Gg = 0 \\ g(0) = 0, \quad g'(0) = 1, \end{cases} \qquad \begin{cases} \bar{g}'' - \bar{G}\bar{g} = 0 \\ \bar{g}(0) = 0, \quad \bar{g}'(0) = 1, \end{cases} \tag{94}$$

Suppose that g *is positive on* $\mathbb{R}^+$*, and that* $g^{-2} \in L^1(+\infty)$*. If*

$$G(r) - \bar{G}(r) \leq \chi_{g^2}(r), \qquad \frac{G(r) - \bar{G}(r)}{\sqrt{\chi_{g^2}(r)}} \in L^1(+\infty), \tag{95}$$

Then $\bar{g} > 0$ *on* $\mathbb{R}^+$ *and* $\bar{g}(r) \sim Cg(r)$ *as* $r \to +\infty$*, for some* $C > 0$.

Proof. Setting $h = \bar{g}/g$, h solves

$$\begin{cases} (vh')' + Avh = 0 \\ h(0) = 1, \qquad h'(0) = 0 \end{cases} \qquad \text{where} \qquad v(r) = g(r)^2, \ A(r) = G(r) - \bar{G}(r).$$

Now, we apply Theorem 5 with the choice $k = 0$. Note that the first requirement in (95) is equivalent to $A \leq \chi$, whence $h > 0$ on $\mathbb{R}^+$ and thus $\bar{g} > 0$. The conclusion $h(r) \to C > 0$ as $r \to +\infty$ proves the desired asymptotic relation. □

Remark 15. A mild sufficient condition to guarantee that both $g > 0$ on $\mathbb{R}^+$ and $g^{-2} \in L^1(+\infty)$ is given by the inequality

$$G(r) \geq -\frac{1}{4r^2} \qquad \text{on } \mathbb{R}^+. \tag{96}$$

This claim is a consequence of Proposition 1.21 in [1], which enables us to deduce both the positivity of g and the bound $g(r) \geq C\sqrt{r}\log r$, for some constant $C > 0$ and $r >> 1$, whenever (96) is met.

Remark 16. By a way of example, suppose that $G(r) = 0$, that is, that we are in a Euclidean setting. Then, (95) reads

$$\bar{G}(r) \geq -\frac{1}{4r^2} \quad \text{on } \mathbb{R}^+, \qquad r\bar{G}(r) \in L^1(+\infty). \tag{97}$$

On the other hand, in the hyperbolic case $G(r) = H^2 > 0$, by the expression (55) for $\chi_{g^2}(r)$ (which coincides with $\chi_3(r)$) the first formula in (95) translates into

$$\bar{G}(r) \geq H^2 - \frac{H^2}{(1-e^{-2Hr})^2} = -H^2 e^{-2Hr}\frac{2-e^{-2Hr}}{(1-e^{-2Hr})^2} \quad \text{on } \mathbb{R}^+, \tag{98}$$

while the second one is simply

$$\bar{G}(r) - H^2 \in L^1(+\infty).$$

We underline that (98) is a very mild requirement, as it is satisfied, for instance, by every $\bar{G}(r) \geq 0$.

We conclude this section with another estimate for the solution of the linear Cauchy problem. This will be used in the proof of the non-existence result, Theorem 12 below, that complements and prove sharpness of the general existence Theorems 9 and 10. We begin with the following version of Gronwall's inequality

Lemma 1. *Let $\varphi, \psi \in L^1_{\mathrm{loc}}(\mathbb{R}^+_0)$, and let $\eta \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^+_0)$ be a solution of the differential inequality*

$$\eta' \leq \varphi\eta + \psi \qquad \text{on } \mathbb{R}^+. \tag{99}$$

Then,

$$\eta(r) \leq \eta(0)\exp\left\{\int_0^r \varphi(t)\mathrm{d}t\right\} + \int_0^r \psi(t)\exp\left\{\int_t^r \varphi(\xi)\mathrm{d}\xi\right\}\mathrm{d}t \qquad \text{on } \mathbb{R}^+_0. \tag{100}$$

Proof. Observe that, for each $s \in \mathbb{R}^+_0$, by (99)

$$\begin{aligned}\left(\eta(r)\exp\left\{-\int_s^r \varphi(\xi)\mathrm{d}\xi\right\}\right)' &= \exp\left\{-\int_s^r \varphi(\xi)\mathrm{d}\xi\right\}[\eta'(r) - \varphi(r)\eta(r)] \\ &\leq \psi(r)\exp\left\{-\int_s^r \varphi(\xi)\mathrm{d}\xi\right\}.\end{aligned}$$

Integrating this inequality on $[0, r]$ gives

$$\eta(r)\exp\left\{-\int_s^r \varphi(\xi)\mathrm{d}\xi\right\} - \eta(0)\exp\left\{-\int_s^0 \varphi(\xi)\mathrm{d}\xi\right\} \leq \int_0^r \psi(t)\exp\left\{-\int_s^t \varphi(\xi)\mathrm{d}\xi\right\}\mathrm{d}t,$$

from which we immediately deduce the validity of (100) by setting $s = r$. $\square$

Remark 17. In the assumptions of the lemma, if $\eta' \geq -\varphi(r)\eta + \psi(r)$ on $\mathbb{R}^+$, for some $\psi, \varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^+_0)$, then

$$\eta(r) \geq \eta(0)\exp\left\{-\int_0^r \varphi(t)\mathrm{d}t\right\} + \int_0^r \psi(t)\exp\left\{-\int_t^r \varphi(\xi)\mathrm{d}\xi\right\}\mathrm{d}t \qquad \text{on } \mathbb{R}^+_0.$$

Using Lemma 1 we obtain

Proposition 2. *Let M_g be a model of dimension $m \geq 3$ and with $K_{\mathrm{rad}} \leq 0$, and define as usual $v = g^{m-1}$. Let $A \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+_0)$ be and let h be a positive solution of the problem*

$$\begin{cases} (vh')' + Avh = 0 & \text{on } \mathbb{R}^+ \\ h(0) = 1, \quad h'(0) = 0. \end{cases} \tag{101}$$

Then,

$$h(r) \leq \exp\left\{ \frac{1}{m-2} \int_0^r \frac{g(t)}{g'(t)} A_-(t) \mathrm{d}t \right\} \qquad \text{on } \mathbb{R}^+_0 \tag{102}$$

Remark 18.

i) Note that, since $K_{\mathrm{rad}} = -g''/g$, the request that K_{rad} is non-positive is equivalent to assume that $g'' \geq 0$. Consequently, in the above assumption $g'(r) \geq g'(0) = 1$ and the RHS of (102) is well defined.

ii) We stress that we have not required $A(r) \leq \chi(r)$, thus the positivity of h is not guaranteed "a priori".

Proof of Proposition 2. We integrate (101) and we use the initial conditions to see that h satisfies

$$\begin{aligned} h(r) &= 1 + \int_0^r g(t)^{1-m} \left(\int_0^t g^{m-1}(s) \big[-A(s) \big] h(s) \mathrm{d}s \right) \mathrm{d}t \\ &\leq 1 + \int_0^r g(t)^{1-m} \left(\int_0^t g^{m-1}(s) A_-(s) h(s) \mathrm{d}s \right) \mathrm{d}t \end{aligned}$$

on $\mathbb{R}^+_0$. Since $g'' \geq 0$, g' is non-decreasing on $\mathbb{R}^+$, thus integrating by parts we have

$$\begin{aligned} h(r) &= 1 + \int_0^r g'(t) g(t)^{1-m} \left(\int_0^t \frac{g^{m-1}(s)}{g'(t)} A_-(s) h(s) \mathrm{d}s \right) \mathrm{d}t \\ &\leq 1 + \int_0^r g'(t) g(t)^{1-m} \left(\int_0^t \frac{g^{m-1}(s)}{g'(s)} A_-(s) h(s) \mathrm{d}s \right) \mathrm{d}t \\ &= 1 + \left[-\frac{g(t)^{2-m}}{m-2} \int_0^t \frac{g^{m-1}(s)}{g'(s)} A_-(s) h(s) \mathrm{d}s \right]_0^r + \\ &\quad + \int_0^r \frac{h(t)}{m-2} \frac{g(t)}{g'(t)} A_-(t) \mathrm{d}t, \end{aligned}$$

that is,

$$h(r) \leq 1 + \frac{1}{m-2} \int_0^r \frac{g(t)}{g't)} A_-(t) h(t) \mathrm{d}t. \tag{103}$$

We set

$$\eta(r) = \frac{1}{m-2} \int_0^r \frac{g(t)}{g't)} A_-(t) h(t) \mathrm{d}t. \tag{104}$$

Differentiating and using (103) we get

$$\eta' = \frac{1}{m-2} \frac{g'}{g} A_- h \leq \frac{1}{m-2} \frac{g'}{g} A_- + \frac{1}{m-2} \frac{g'}{g} A_- \eta$$

on $\mathbb{R}^+_0$. We observe that $\eta(0)=0$, thus applying Lemma 1 we obtain

$$
\begin{aligned}
h(r) &= 1+\eta(r) \le 1 + \int_0^r \frac{1}{m-2}\frac{g(t)}{g'(t)}A_-(t)\exp\left\{\int_t^r \frac{1}{m-2}\frac{g(\xi)}{g'(\xi)}A_-(\xi)\mathrm{d}\xi\right\}\mathrm{d}t \\
&= 1 - \int_0^r \frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left\{\int_t^r \frac{1}{m-2}\frac{g(\xi)}{g'(\xi)}A_-(\xi)\mathrm{d}\xi\right\}\right)\mathrm{d}t \\
&= \exp\left\{\int_0^r \frac{1}{m-2}\frac{g(\xi)}{g'(\xi)}A_-(\xi)\mathrm{d}\xi\right\},
\end{aligned}
$$

which proves the desired inequality. □

A first insight: generalizing Ni's result

In this section, we show a first existence theorem for Yamabe type equations on model manifolds, recovering, as a side product, Ni's result quoted in the Introduction. Its proof will be accomplished via a generalization of the arguments in [24] in order to deal with the linear term. However, we underline that next results, on non-Euclidean spaces, seem to be hardly obtainable from Ni's original approach, which is based on the construction of some special functions suitable only for $\mathbb{R}^m$. In carefully revising Ni's technique, we give a first glance into the core of the problems, and devise some of the new strategies that will be presented in the next section of the paper, when we will deal with the most general case. We first need the following

Lemma 2. *Let M_g be a model, set $v = g^{m-1}$ and suppose that (V_{L1}) is met. Let $A \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+_0)$ be such that $A < k\chi$ on $\mathbb{R}^+$, for some $k \in (-\infty, 1]$. Let $B \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+_0)$, $B \ge 0$, and $\sigma > 1$. For each fixed $\alpha > 0$, consider a* $\mathrm{Lip}_{\mathrm{loc}}$ *solution z_α of*

$$
\begin{cases}
(vz_\alpha')' + Avz_\alpha + Bv|z_\alpha|^{\sigma-1}z_\alpha = 0 & \text{on } [0,\varepsilon_\alpha) \\
z_\alpha(0) = \alpha > 0, \qquad z_\alpha'(0) = 0,
\end{cases}
\tag{105}
$$

for some $\varepsilon_\alpha > 0$. Let h and $\bar{h}$ be positive solutions of

$$
\begin{cases}
(vh')' + Avh \ge 0 \\
h(0) = 1, \qquad h'(0) = 0
\end{cases}, \qquad
\begin{cases}
(v\bar{h}')' + Av\bar{h} \le 0 \\
\bar{h}(0) = 1, \qquad \bar{h}'(0) = 0
\end{cases}
\tag{106}
$$

on $\mathbb{R}^+_0$. Suppose that $B(r)$ satisfies

$$
B(r) \le C\frac{k\chi(r) - A(r)}{h(r)^\sigma}H_k(r) \tag{107}
$$

for $r \ge r_1$, sufficiently large and for some constant $C > 0$, the function $H_k(r)$ being defined in (58). Then, there exists $\alpha_0 > 0$ such that, for every $\alpha \in (0, \alpha_0)$, z_α can be extended to a positive, locally Lipschitz solution on $\mathbb{R}^+_0$ of (105) satisfying

$$
\frac{\alpha}{2}\bar{h}(r) \le z_\alpha(r) \le \alpha h(r) \qquad \text{on } \mathbb{R}^+_0. \tag{108}
$$

Moreover, if $A \ge 0$, then $z_\alpha' \le 0$ on $\mathbb{R}^+$.

Remark 19. The local existence for (105) is achieved, for instance, via Picard iteration procedure or a modification of Proposition 4.3 in [1]. This last argument also shows that $z'_\alpha(0) = 0$, and positivity follows from the initial data and continuity. Furthermore, since $A \le \chi$, Theorem 4 guarantees the existence of positive solutions $h, \bar{h}$ of (106).

Proof of Lemma 2. On the interval $[0, \varepsilon_\alpha)$ we consider the function $\xi = z_\alpha/h$. From (105), (106) and $B \ge 0$ we deduce

$$\begin{cases} (h^2 v\xi')' \le 0 & \text{on } (0, \varepsilon_\alpha) \\ \xi(0) = \alpha, \qquad \xi'(0) = 0. \end{cases}$$

Integrating we obtain $\xi' \le 0$, and therefore $\xi(r) \le \xi(0) = \alpha$. In other words,

$$z_\alpha(r) \le \alpha h(r) \qquad \text{on } [0, \varepsilon_\alpha), \tag{109}$$

We now look for a lower bound of z_α on $[0, \varepsilon_\alpha)$. Towards this aim we observe that, since $A(r) < k\chi(r)$ on $\mathbb{R}^+_0$, we can define $\widetilde{A} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^+_0)$ in such a way that

$$A < \widetilde{A} \le k\chi \quad \text{on } \mathbb{R}^+, \qquad \widetilde{A} \equiv k\chi \quad \text{on } [r_1, +\infty),$$

for some $r_1 >> 1$. Next, let w be the solution of

$$\begin{cases} (vw')' + \widetilde{A}vw = 0 & \text{on } \mathbb{R}^+ \\ w(0) = 1, \qquad w'(0) = 0. \end{cases} \tag{110}$$

Then, by Theorem 4, $w > 0$ on $\mathbb{R}^+$ and it satisfies the estimate

$$w(r) \sim CH_k(r) \qquad \text{as } r \to +\infty \tag{111}$$

for some constant $C > 0$. Hence, using assumption (107) we deduce the existence of a constant $C_1 > 0$ sufficiently large such that

$$B(r) \le C_1 \frac{\big[\widetilde{A}(r) - A(r)\big] w(r)}{h(r)^\sigma} \qquad \text{on } \mathbb{R}^+_0. \tag{112}$$

Note that, for the existence of C_1, it is necessary that $A < \widetilde{A}$ on $\mathbb{R}^+$ and therefore that the strict inequality $A < k\chi$ holds.

For any $\beta > 0$ we set $w_\beta = \beta w$, and consider $\varphi = z_\alpha - w_{\alpha/2}$. Then, by the initial conditions for z_α and $w_{\alpha/2}$, $\varphi > 0$ on some maximal interval $[0, \bar{\varepsilon}_\alpha) \subseteq [0, \varepsilon_\alpha)$. From (105), (110), (112) and (109) we get

$$\begin{aligned} (v\varphi')' &= -Bv|z_\alpha|^\sigma - Avz_\alpha + \widetilde{A}vw_{\alpha/2} \\ &= -Av\varphi + (\widetilde{A} - A)vw_{\alpha/2} - Bv|z_\alpha|^\sigma \\ &\ge -Av\varphi + (\widetilde{A} - A)vw_{\alpha/2} - C_1 v \frac{(\widetilde{A} - A)w}{h^\sigma}(\alpha h)^\sigma \\ &= -Av\varphi + (\widetilde{A} - A)vw_{\alpha/2}\left(1 - 2C_1\alpha^{\sigma-1}\right), \end{aligned} \tag{113}$$

hence $(v\varphi')' + Av\varphi \ge 0$ on $[0, \bar{\varepsilon}_\alpha)$ provided $\alpha \le (2C_1)^{-1/(\sigma-1)} = \alpha_0$. Next, on $[0, \bar{\varepsilon}_\alpha)$ we define $\eta = \varphi/\bar{h}$. Using (106) and $\varphi > 0$ we get

$$\begin{cases} (\bar{h}^2 v\eta')' \ge 0 & \text{on } (0, \bar{\varepsilon}_\alpha) \\ \eta(0) = \alpha/2, \qquad \eta'(0) = 0. \end{cases}$$

Integrating, we have $\eta' \geq 0$, hence from $\eta(r) \geq \eta(0) = \alpha/2$ we finally get $\eta \geq \alpha/2$. Since, by construction, $\eta(\bar{\varepsilon}_\alpha) = 0$ whenever $\bar{\varepsilon}_\alpha < \varepsilon_\alpha$, we deduce that necessarily $\bar{\varepsilon}_\alpha = \varepsilon_\alpha$ and thus

$$z_\alpha \geq w_{\alpha/2} + \frac{\alpha}{2}\bar{h} \qquad \text{on } [0, \varepsilon_\alpha), \tag{114}$$

completing the proof of (108) restricted to $[0, \varepsilon_\alpha)$. Since $h, \bar{h}, w$ are defined and positive on $\mathbb{R}^+$, z_α cannot explode in a finite time and can therefore be extended to a positive solution on the whole $\mathbb{R}^+$.

We are left to prove that $z'_\alpha \leq 0$ whenever $A \geq 0$. This follows immediately from a first integration of (105), recalling that $B \geq 0$ and $z_\alpha > 0$ on $\mathbb{R}^+$. □

Remark 20. Following the argument of the proof we see that C_1, and therefore α_0, depends on our choice of r_1. We also observe that, without loss of generality, in the above proof we could have taken $h = \bar{h} = y$ solution of $(vy')' + Avy = 0$ with initial condition $y(0) = 1$, $y'(0) = 0$. Indeed, from Sturm-type arguments, $\bar{h} \leq y \leq h$, and hence (107) implies

$$B(r) \leq C\frac{k\chi(r) - A(r)}{y(r)^\sigma} H_k(r), \tag{115}$$

and z_α turns out to satisfy the more stringent condition $\alpha/2y \leq z_\alpha \leq \alpha y$. We have preferred to keep $h, \bar{h}$ distinct since condition (107) only needs an explicit h that solves a differential inequality.

Remark 21. Suppose that $h = \bar{h} = y$ as in the above remark, and assume that

$$\frac{k\chi(r) - A(r)}{\sqrt{\chi_{H_k^2 v}(r)}} \in L^1(+\infty), \tag{116}$$

for some $k \in (-\infty, 1]$. Then, applying Theorem 5 we infer the asymptotic relation $y(r) \sim CH_k(r)$ as r diverges, for some $C > 0$. Therefore, condition (115) is equivalent to

$$B(r) \leq C\frac{k\chi(r) - A(r)}{H_k(r)^{\sigma-1}} \qquad \text{for } r >> 1. \tag{117}$$

We stress that the above condition implies

$$0 \leq \frac{B(r)H_k(r)^{\sigma-1}}{\sqrt{\chi_{H_k^2 v}(r)}} \leq C\frac{k\chi(r) - A(r)}{\sqrt{\chi_{H_k^2 v}(r)}} \in L^1(+\infty). \tag{118}$$

This enables us to compare, in the specific case when (116) is met, Lemma 2 with the next Theorem 7

Remark 22. It is reasonable to question whether the restriction of the initial condition α to $(0, \alpha_0)$, for the desired solution of (105), is of a technical nature or properly pertains to the setting we are considering. As a matter of fact, it is essential for the existence of a global positive solution. Indeed, in Theorem B of [32] it is shown that, for $v(r) = r^{m-1}$ and $m \geq 3$, any solution z_α of

$$\begin{cases} (r^{m-1}z'_\alpha)' + Ar^{m-1}z_\alpha + Br^{m-1}|z_\alpha|^{\sigma-1}z_\alpha = 0 & \text{on } \mathbb{R}^+, \\ z_\alpha(0) = \alpha > 0, \qquad z'_\alpha(0) = 0, \end{cases} \tag{119}$$

starting from a sufficiently large α, has necessarily a first zero whenever

$$\begin{cases} 1 < \sigma < \dfrac{m+2}{m-2}; \\ -\dfrac{C^2}{r^{2-\varepsilon}} \leq A(r) \leq 0 \qquad \text{for some} \quad C > 0, \quad \varepsilon \in (0, m-2); \\ B \in C^0(\mathbb{R}_0^+), \qquad B \geq 0, \qquad B(0) > 0. \end{cases}$$

The interested reader can also consult related results in [13]. It seems interesting to note that the technique employed in [32], [13] is heavily based on the fact that the volume $v(r)$ is polynomial, so that the solution of (119) is "close" to the one of

$$\begin{cases} (r^{m-1}\bar{z}_\alpha')' + B(0)r^{m-1}|\bar{z}_\alpha|^{\sigma-1}\bar{z}_\alpha = 0 \qquad \text{on } \mathbb{R}^+, \\ \bar{z}_\alpha(0) = \alpha > 0, \qquad \bar{z}_\alpha'(0) = 0, \end{cases} \tag{120}$$

and that this latter has a first zero. It would be very interesting to investigate on analogous results for general growth of $v(r)$.

With the aid of Lemma 2 we now prove

Theorem 6. *Let M_g be an m-dimensional model manifold, set $v = g^{m-1}$ and suppose that (V_{L1}) is met. Let $\sigma > 1$, and let $A(r(x)) \in \text{Höl}_{\text{loc}}(M_g)$ be a radial function satisfying*

$$A(r) < k\chi(r) \quad \text{on } \mathbb{R}^+,$$

for some $k \in (-\infty, 1]$. Let $h, \bar{h}$ be positive, C^2 solutions of (106) on $\mathbb{R}^+$. Consider a function $b(x) \in \text{Höl}_{\text{loc}}(M_g)$ satisfying

$$|b(x)| \leq C \frac{k\chi\big(r(x)\big) - A\big(r(x)\big)}{h\big(r(x)\big)^\sigma} H_k\big(r(x)\big) \tag{121}$$

outside some ball and for some constant $C > 0$. Then, the equation

$$\Delta u + A(r(x))u - b(x)u^\sigma = 0 \tag{122}$$

possesses infinitely many solutions $\{u_j\}_{j\in\mathbb{N}} \subseteq \text{Höl}^2_{\text{loc}}(M_g)$. For each of them, there exist constants $0 < \Gamma_{1,j} \leq \Gamma_{2,j}$ such that

$$\Gamma_{1,j}\bar{h}(r(x)) \leq u_j(x) \leq \Gamma_{2,j}h(r(x)) \qquad \text{on } M_g. \tag{123}$$

Furthermore, $\Gamma_{2,j} \downarrow 0$ as $j \to +\infty$. If $A(r)$ and $b \in C^\infty(M_g)$, then $\{u_j\} \subseteq C^\infty(M_g)$.

Proof. First of all we prove the theorem in case $h = \bar{h} = y$ is a solution of

$$\begin{cases} (vy')' + Avy = 0 \\ y(0) = 1 \qquad y'(0) = 0. \end{cases} \tag{124}$$

Next we choose a function $B(r) \geq 0$ on $\mathbb{R}_0^+$, $B \in L^\infty_{\text{loc}}(\mathbb{R}_0^+)$ such that $|b(x)| \leq B(r(x))$ on M_g and satisfying (107). This is possible because of (121). By Lemma 2, there exists $\alpha_0 > 0$ such that for each $\alpha \in (0, \alpha_0)$ we have a positive solution z_α on $\mathbb{R}_0^+$ of (105). Setting $\omega_\alpha(x) = z_\alpha(r(x))$, ω_α solves

$$\begin{aligned} \Delta\omega_\alpha + A\omega_\alpha - b\omega_\alpha^\sigma &\leq \Delta\omega_\alpha + A\omega_\alpha + B\omega_\alpha^\sigma \\ &= z_\alpha'' + \frac{v'}{v}z_\alpha' + Az_\alpha + Bz_\alpha^\sigma \\ &= v^{-1}\big[(vz_\alpha')' + Avz_\alpha + Bvz_\alpha^\sigma\big] = 0 \end{aligned}$$

and has the property that

$$\frac{\alpha}{2}y(r(x)) \le \omega_\alpha(x) \le \alpha y(r(x)) \tag{125}$$

on M_g. Next, we define

$$y_\alpha = \frac{\alpha}{2}y - z_{\alpha/4}.$$

Then, $y_\alpha(0) = \alpha/4$ and, by (108),

$$\frac{\alpha}{2}y \ge y_\alpha = \frac{\alpha}{2}y - z_{\alpha/4} \ge \left(\frac{\alpha}{2} - \frac{\alpha}{4}\right) y = \frac{\alpha}{4}y.$$

Therefore, using (125),

$$\frac{\alpha}{4}y \le y_\alpha \le \frac{\alpha}{2}y \le \omega_\alpha \le \alpha y.$$

Furthermore,

$$(vy_\alpha')' + Avy_\alpha = Bvz_{\alpha/4}^\sigma = Bv\left(\frac{z_{\alpha/4}}{\frac{\alpha}{2}y - z_{\alpha/4}}\right)^\sigma y_\alpha^\sigma.$$

Since

$$\frac{z_{\alpha/4}}{\frac{\alpha}{2}y - z_{\alpha/4}} \ge \frac{\frac{\alpha}{8}y}{\frac{\alpha}{2}y} = \frac{1}{4},$$

it follows that y_α solves

$$(vy_\alpha')' + Avy_\alpha \ge Bv4^{-\sigma}y_\alpha^\sigma,$$

whence, defining $\bar{y}_\alpha = 4^{-\frac{\sigma}{\sigma-1}}y_\alpha$,

$$(v\bar{y}_\alpha')' + Av\bar{y}_\alpha \ge Bv\bar{y}_\alpha^\sigma.$$

As a consequence, $\bar{\omega}_\alpha(x) = \bar{y}_\alpha(r(x))$ satisfies

$$\Delta\bar{\omega}_\alpha + A\bar{\omega}_\alpha - b\bar{\omega}_\alpha^\sigma \ge (B-b)\bar{\omega}_\alpha^\sigma \ge 0 \tag{126}$$

and

$$4^{\frac{-\sigma}{\sigma-1}}\frac{\alpha}{4}y(r(x)) \le \bar{\omega}_\alpha(x) \le 4^{\frac{-\sigma}{\sigma-1}}\frac{\alpha}{2}y(r(x)) \le \frac{\alpha}{2}y(r(x)) \le \omega_\alpha(x) \tag{127}$$

on M_g. By the monotone iteration scheme, [33], and elliptic regularity, there exists a solution $u_\alpha(x) \in \text{Höl}^2_{\text{loc}}(M_g)$ of (122) satisfying

$$\bar{\omega}_\alpha \le u_\alpha \le \omega_\alpha \tag{128}$$

on M_g. Furthermore, if $A(r), b$ are smooth, then u_α is smooth again by elliptic regularity. From (127) and (128) it follows immediately that

$$4^{-\frac{2\sigma+1}{\sigma-1}}\alpha y(r(x)) \le u_\alpha(x) \le \alpha y(r(x))$$

The procedure can now be iterated, simply replacing $\alpha = \alpha_1$ with, say, $\alpha_2 = 4^{-\frac{2\sigma+1}{\sigma-1}-1}\alpha$. Note that the corresponding positive solution u_{α_2} is strictly below $u_{\alpha_1} = u_\alpha$. In this way we obtain the required conclusion. If $h \not\equiv \bar{h}$, we reason as in Remark 20. Let y be a solution of (124). Then, by Sturm comparison $\bar{h} \le y \le h$, thus the validity of (121) implies the validity of

$$|b(x)| \le C\frac{k\chi(r(x)) - A(r(x))}{y(r(x))^\sigma}H_k(r(x)).$$

Applying the previous proof we get a sequence of solutions u_j such that

$$\Gamma_{1,j}\bar{h}(r(x)) \le \Gamma_{1,j}y(r(x)) \le u_j(x) \le \Gamma_{2,j}y(r(x)) \le \Gamma_{2,j}h(r(x))$$

on M_g. This completes the proof of the theorem. $\square$

To better appreciate the above result, we specialize it to the Yamabe problem on $\mathbb{R}^m$ to give an alternative proof of Ni's version of Theorem 1.

Corollary 5 ([24], Theorem 1.4)**.** *Consider the Euclidean space $\mathbb{R}^m$, $m \ge 3$, and let $\widetilde{s}(x) \in C^\infty(\mathbb{R}^m)$ be a function satisfying*

$$|\widetilde{s}(x)| \le \frac{C}{r(x)^l} \qquad \text{for } r(x) \ge 1, \tag{129}$$

for some $C > 0$, $l > 2$. Then, the Euclidean metric $\langle\, ,\rangle$ can be conformally deformed to a complete, smooth metric $\widetilde{\langle\, ,\rangle}$ of scalar curvature $\widetilde{s}(x)$ and satisfying

$$\Gamma_1\langle\, ,\rangle_x \le \widetilde{\langle\, ,\rangle}_x \le \Gamma_2\langle\, ,\rangle_x \qquad \forall\, x \in \mathbb{R}^m, \tag{130}$$

for some $0 < \Gamma_1 \le \Gamma_2$. Furthermore, Γ_2 can be chosen to be as small as we wish.

Proof. Defining u as in (2), u must be a positive solution of (3), which on Euclidean space reads

$$\Delta u + \frac{\widetilde{s}(x)}{c_m}u^{\frac{m+2}{m-2}} = 0, \qquad \text{where } \ c_m = \frac{4(m-1)}{m-2}.$$

Set $\sigma = (m+2)/(m-2)$, $b(x) = -\widetilde{s}(x)/c_m$, and realize $\mathbb{R}^m$ as a model manifold with $g(r) = r$, for which $\chi(r) = (m-2)^2/(4r^2)$. To apply Theorem 6, we choose $A \equiv 0$, $h = \bar{h} = 1$ and $k > 0$ small enough in such a way that

$$2 + (m-2)\frac{1-\sqrt{1-k}}{2} < l.$$

This is possible since $l > 2$. Then, $|b(x)| \le Cr^{-l}$ implies the inequality

$$|b(x)| \le Cr(x)^{-2-\frac{m-2}{2}(1-\sqrt{1-k})}, \tag{131}$$

which is (121) in our setting. Thus, the existence of the desired conformal deformations follows from Theorem 6. $\square$

The main existence theorems

We begin with the next theorem, that shall be compared with the previous Lemma 2. Our aim is to construct positive solutions of (105) with a precise asymptotic behaviour at infinity. This will be achieved by requiring some integrability condition on $B(r)$ playing the role of (121).

Theorem 7. *Let $g \in C^2(\mathbb{R}^+_0)$, $m \ge 2$ be such that $v = g^{m-1}$ satisfies (V_{L1}). Let $A \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+_0)$ with the property that $A(r) \le \chi(r)$, and let $h \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^+_0)$ be the positive solution of*

$$\begin{cases} (vh')' + Avh = 0 & \text{on } \mathbb{R}^+ \\ h(0) = 1, \qquad h'(0) = 0. \end{cases} \tag{132}$$

Let $\sigma > 1$, $B \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+_0)$ and suppose that

$$\frac{B(r)h(r)^{\sigma-1}}{\sqrt{\chi_{h^2v}(r)}} \in L^1(+\infty). \tag{133}$$

Then, there exists a constant $\beta > 0$, depending on σ, g, A, B, h such that the following holds: for each $\gamma_\infty \in (0, \beta)$, there exist $0 < \gamma_0 \le \gamma_M$ and a positive solution γ of

$$\begin{cases} (v\gamma')' + Av\gamma = Bv\gamma^\sigma & \text{on } \mathbb{R}^+ \\ \gamma(0) = \gamma_0, \quad \gamma'(0) = 0 \end{cases} \tag{134}$$

such that

$$\gamma(r) \le \gamma_M h(r) \quad \text{on } \mathbb{R}^+, \qquad \frac{\gamma(r)}{h(r)} \to \gamma_\infty \qquad \text{as } r \to +\infty. \tag{135}$$

Moreover, $\gamma_M \to 0$ as $\gamma_\infty \to 0$.

Remark 23. Condition (133) is well defined provided

$$\frac{1}{h^2 v} \in L^1(+\infty), \tag{136}$$

which follows, in our assumptions, from Remark 11.

Proof of Theorem 7. The idea is the same as that of Proposition 1, and relies on a squeezing method. Due to the presence of some technical details, we prefer to write the proof carefully for the convenience of the reader.

Let h be the solution of (132) and note that, since $A \le \chi$, h is indeed positive on $\mathbb{R}^+_0$ by Theorem 4. If γ solves (134) then $\xi = \gamma/h$ is a solution of

$$\begin{cases} (h^2 v\xi')' = (Bh^{\sigma-1})h^2 v\xi^\sigma & \text{on } \mathbb{R}^+ \\ \xi(0) = \gamma_0, \quad \xi'(0) = 0 \end{cases} \tag{137}$$

Define

$$\tilde{v}(r) = h(r)^2 v(r),$$

and note that, by its very definition, $\chi_{\tilde{v}}(r) \to +\infty$ as $r \to 0^+$. Using (133) and this observation we set Λ_+, Λ_- according to

$$\Lambda_\pm = \frac{1}{2}\int_0^{+\infty} \frac{B_\pm(r)h(r)^{\sigma-1}}{\sqrt{\chi_{\tilde{v}}(r)}}\mathrm{d}r. \tag{138}$$

We perform the change of variables (72), ξ is a solution of (137) if and only if $y(s) = s\xi(r(s))$ solves

$$\begin{cases} \ddot{y}(s) = \dfrac{\tilde{v}^2(r(s))}{s^3} B(r(s))h(r(s))^{\sigma-1}\left(\dfrac{y(s)}{s}\right)^\sigma & \text{on } \mathbb{R}^+. \\ y(0) = 0, \quad \dot{y}(0) = \gamma_0, \end{cases} \tag{139}$$

where $\cdot$ means the derivative with respect to s. Set $\Lambda = \max\{\Lambda_+, \Lambda_-\}$, and define

$$\beta = \min\left\{(\Lambda\sigma)^{-\frac{1}{\sigma-1}}, \sup\left\{t - t^\sigma\Lambda \;:\; t \in \left[0, (\Lambda\sigma)^{-\frac{1}{\sigma-1}}\right]\right\}\right\}, \tag{140}$$

It is easy to check that, for each $\gamma_\infty \in (0, \beta)$, we can choose in a unique way

$$\gamma_M \in \left(0, (\Lambda\sigma)^{-\frac{1}{\sigma-1}}\right)$$

in such a way that

$$\gamma_M - \gamma_M^\sigma \Lambda_- = \gamma_\infty. \tag{141}$$

From this and (140) we deduce

$$\gamma_\infty \leq \gamma_M \qquad \text{and} \qquad \gamma_\infty - (\gamma_\infty)^\sigma \Lambda_+ \geq \gamma_\infty - (\gamma_\infty)^\sigma \Lambda > 0. \tag{142}$$

We can thus define γ_m accordingly to

$$0 < \gamma_m = \gamma_\infty - (\gamma_\infty)^\sigma \Lambda_+ \leq \gamma_\infty \leq \gamma_M. \tag{143}$$

Next, we consider the problem

$$\begin{cases} \ddot{y}(s) = -\dfrac{\tilde{v}^2(r(s))}{s^3} B_-(r(s)) h(r(s))^{\sigma-1} \gamma_M^\sigma & \text{on } \mathbb{R}^+ \\ y(0) = 0, \quad \dot{y}(0) = \gamma_M. \end{cases} \tag{144}$$

Integrating the equation in (144) once, changing variables using (72) and recalling (73), (141) we obtain

$$\begin{aligned} \dot{y}(s) &= \gamma_M - \gamma_M^\sigma \int_0^{r(s)} \tilde{v}(\rho) \frac{B_-(\rho) h(\rho)^{\sigma-1}}{s(\rho)} \mathrm{d}\rho \\ &= \gamma_M - \gamma_M^\sigma \int_0^{r(s)} B_-(\rho) h(\rho)^{\sigma-1} \tilde{v}(\rho) \left[\int_\rho^{+\infty} \frac{\mathrm{d}\tau}{\tilde{v}(\tau)}\right] \mathrm{d}\rho \\ &= \gamma_M - \frac{\gamma_M^\sigma}{2} \int_0^{r(s)} \frac{B_-(\rho) h(\rho)^{\sigma-1}}{\sqrt{\chi_{\tilde{v}}(\rho)}} \mathrm{d}\rho. \end{aligned} \tag{145}$$

Therefore, by (138),

$$\dot{y}(s) \geq \gamma_M - \gamma_M^\sigma \Lambda_- = \gamma_\infty > 0;$$

moreover, since $r(s) \to +\infty$ as $s \to +\infty$,

$$\lim_{s \to +\infty} \dot{y}(s) = \gamma_M - \gamma_M^\sigma \int_0^{+\infty} \frac{B_-(\rho) h(\rho)^{\sigma-1}}{\sqrt{\chi_{\tilde{v}}(\rho)}} \mathrm{d}\rho = \gamma_M - \gamma_M^\sigma \Lambda_- = \gamma_\infty. \tag{146}$$

The function y is thus increasing and, since $y(0) = 0$, y is positive on $\mathbb{R}^+$. From (145) we also deduce

$$\gamma_\infty \leq \dot{y}(s) \leq \gamma_M \qquad \text{on } \mathbb{R}^+. \tag{147}$$

Integrating (147) on $[0, s]$ and using $y(0) = 0$ we have

$$\gamma_\infty s \leq y(s) \leq \gamma_M s \qquad \text{on } \mathbb{R}^+ \tag{148}$$

and, from (146) and De l'Hopital's rule,

$$\lim_{s \to +\infty} \frac{y(s)}{s} = \gamma_\infty. \tag{149}$$

We define

$$\xi_+(r) = \frac{y(s(r))}{s(r)}. \tag{150}$$

Then, ξ_+ is positive on $\mathbb{R}^+$, and

$$\xi_+'(r) = \frac{s'(r)}{s^2(r)}\Big\{\dot{y}(s(r))s(r) - y(s(r))\Big\} = \frac{s'(r)}{s^2(r)}\mu(s(r)).$$

Studying the function $\mu(s) = s\dot{y}(s) - y(s)$ we have $\mu(0) = 0$ and $\dot{\mu}(s) = s\ddot{y}(s) \leq 0$ on $\mathbb{R}^+$. It follows that ξ_+ is non-increasing on $\mathbb{R}^+$. Furthermore, $\xi_+(0^+) = \gamma_M$ and, because of (150) and (149), $\xi_+(r) \downarrow \gamma_\infty$ as $r \to +\infty$. From the change of variables (72), from (148) and from the fact that ξ_+ is positive, it also follows that ξ_+ satisfies

$$\begin{cases} (\tilde{v}\xi_+')' = -(B_-h^{\sigma-1})\tilde{v}\gamma_M^\sigma \leq -(B_-h^{\sigma-1})\tilde{v}\xi_+^\sigma \leq (Bh^{\sigma-1})\tilde{v}\xi_+^\sigma & \text{on } \mathbb{R}^+ \\ \xi_+(0) = \gamma_M, \quad \xi_+'(0) = 0, \end{cases}$$

Thus ξ_+ is a supersolution of (137). To construct a subsolution we proceed in a similar way. Consider the problem

$$\begin{cases} \ddot{y}(s) = \dfrac{\tilde{v}^2(r(s))}{s^3}B_+(r(s))h(r(s))^{\sigma-1}\gamma_\infty^\sigma & \text{on } \mathbb{R}^+ \\ y(0) = 0, \quad \dot{y}(0) = \gamma_m. \end{cases} \tag{151}$$

Integrating the equation on $[0, s]$, proceeding as in (145) and using (143) we get

$$\dot{y}(s) = \gamma_m + \frac{\gamma_\infty^\sigma}{2}\int_0^{r(s)} \frac{B_+(\rho)h(\rho)^{\sigma-1}}{\sqrt{\chi_{\tilde{v}}(\rho)}}\mathrm{d}\rho \leq \gamma_m + \gamma_\infty^\sigma\Lambda_+ = \gamma_\infty,$$

and $\dot{y}(s) \to \gamma_\infty$ as $s \to +\infty$ from which we deduce

$$\lim_{s\to+\infty} \frac{y(s)}{s} = \gamma_\infty. \tag{152}$$

From $\dot{y}(0) = \gamma_m$, and since $\dot{y}$ is increasing being $\ddot{y} \geq 0$, we argue that

$$\gamma_m \leq \dot{y}(s) \leq \gamma_m + \gamma_\infty^\sigma\Lambda_+ = \gamma_\infty,$$

whence integrating and using $y(0) = 0$ we deduce

$$\gamma_m s \leq y(s) \leq \gamma_\infty s.$$

We define $\xi_-(r) = y(s(r))/s(r)$. Similarly to what we did for the supersolution ξ_+, ξ_- is positive, non-decreasing, $\xi_-(0) = \gamma_m$ and, because of (152), $\xi_-(r) \uparrow \gamma_\infty$ as $r \to +\infty$. From our change of variables, ξ_- satisfies

$$\begin{cases} (\tilde{v}\xi_-')' = (B_+h^{\sigma-1})\tilde{v}\gamma_\infty^\sigma \geq (B_+h^{\sigma-1})\tilde{v}\xi_-^\sigma \geq (Bh^{\sigma-1})\tilde{v}\xi_-^\sigma & \text{on } \mathbb{R}^+ \\ \xi_-(0) = \gamma_m, \quad \xi_-'(0) = 0. \end{cases}$$

The function ξ_- is thus the desired subsolution of (137). Furthermore,

$$\gamma_m \leq \xi_-(r) \leq \gamma_\infty \leq \xi_+(r) \leq \gamma_M, \qquad \xi_-(r), \xi_+(r) \to \gamma_\infty \quad \text{as } r \to +\infty. \tag{153}$$

By the monotone iteration scheme, [33], there exists a radial solution ξ of (137), for some $\gamma_0 \in [\gamma_m, \gamma_M]$, such that

$$\xi_- \le \xi \le \xi_+.$$

In particular,

$$\xi(r) \le \gamma_M, \qquad \xi(r) \to \gamma_\infty \qquad \text{as } r \to +\infty.$$

The fact that $\gamma_M \to 0$ as $\gamma_\infty \to 0$ follows from its very definition (141). To conclude, from $h(0) = 1$, there exists $\gamma_0 \in [\gamma_m, \gamma_M]$ such that the function $\gamma = \xi h$ is a solution of (134) such that

$$\gamma(r) \le \gamma_M h(r) \quad \text{on } \mathbb{R}^+, \qquad \lim_{r \to +\infty} \frac{\gamma(r)}{h(r)} = \gamma_\infty,$$

which proves (169). □

Remark 24. If A is subjected to the requirement

$$\frac{A(r) - k\chi(r)}{\sqrt{\chi_{H_k^2 v}(r)}} \in L^1(+\infty), \tag{154}$$

for some $k \in (-\infty, 1]$, then by Theorem 5 it holds $h(r) \sim CH_k(r)$ as r diverges, for some $C > 0$. Whence, condition (133) on B can be rewritten as

$$\frac{B(r)H_k(r)^{\sigma-1}}{\sqrt{\chi_{H_k^2 v}(r)}} \in L^1(+\infty). \tag{155}$$

Referring to what stated in Remark 21, in the case when (154) is satisfied we deduce that Theorem 7 refines Lemma 2.

The above theorem gives the building block to construct sub- and supersolution for the Yamabe equation, and the effectiveness of condition (133) on B depends on knowing the precise asymptotic behaviour for $h(r)$. This will be accomplished via Theorem 5, under the condition that A is closed to $k\chi$ in the integral sense given by (154). Passing from models to general manifolds requires a control on the behaviour of the Laplacian of the distance function, which is obtained via the classical Laplacian comparison theorems (see [27], Section 2 or [1], Theorems 1.17 and 1.19)). To state them in a way convenient for our purposes, we introduce the next quantities. We restrict ourselves to the case when M has a pole. For $x \in M \backslash \{o\}$, let $c : [0, r(x)] \to M$ be the unique minimizing geodesic connecting o to x. Set

$$\psi_c(s) = s \exp\left\{ \int_0^s \left[\frac{\Delta r \circ c(\sigma)}{m-1} - \frac{1}{\sigma} \right] \mathrm{d}\sigma \right\}, \qquad w_c(s) = \psi_c(s)^{m-1}, \tag{156}$$

and note that

$$\Delta r\big(c(s)\big) = (m-1)\frac{\psi_c'(s)}{\psi_c(s)} = \frac{w_c'(s)}{w_c(s)} \qquad \forall\, s \in [0, r(x)], \tag{157}$$

thus, for every radial $u(x) = \gamma(r(x))$ it holds

$$\Delta u(x) = \gamma'' + \gamma' \Delta r(x) = \left(\gamma'' + \gamma' \frac{w_c'}{w_c} \right)(r(x)) = \big[w_c^{-1}(w_c \gamma')' \big](r(x)). \tag{158}$$

In the notation above (a particular version of) the Laplacian comparison theorems reads as follows:

Theorem 8 ([27], Section 2 and [1], Theorems 1.17 and 1.19)**.** *Let $(M, \langle\, , \rangle)$ be an m-dimensional manifold with a pole o and radial sectional curvature with respect to o satisfying*

$$K_{\mathrm{rad}}(x) \le -G\big(r(x)\big) \qquad \Big(\textit{resp.} \quad K_{\mathrm{rad}}(x) \ge -\bar{G}\big(r(x)\big)\Big),$$

for some $G, \bar{G} \in C^0(\mathbb{R}^+_0)$. Let g (resp. $\bar{g}$) be a solution of

$$\begin{cases} g'' - Gg \le 0 & \text{on } \mathbb{R}^+ \\ g(0) = 0, & g'(0) = 1. \end{cases} \qquad \left(\textit{resp.} \quad \begin{cases} \bar{g}'' - \bar{G}\bar{g} \ge 0 & \text{on } \mathbb{R}^+ \\ \bar{g}(0) = 0, & \bar{g}'(0) = 1. \end{cases}\right)$$

Suppose that g (resp. $\bar{g}$) is positive on $\mathbb{R}^+$, and set $v = g^{m-1}$ (resp. $\bar{v} = \bar{g}^{m-1}$). Then, for every $x \in M \backslash \{o\}$ and every minimizing geodesic $c : [0, r(x)] \to M$ joining o to x,

$$\frac{w_c'}{w_c}\big(r(x)\big) \ge \frac{v'}{v}\big(r(x)\big). \qquad \left(\textit{resp.} \ \frac{w_c'}{w_c}\big(r(x)\big) \le \frac{\bar{v}'}{\bar{v}}\big(r(x)\big)\right)$$

As underlined in the Introduction, the method to produce sub-and supersolution radically depends on whether $k = 0$ or $k > 0$ in (154).

The case $k = 0$.

In this section, we prove Theorem 3 and the subsequent Corollary 2 in the Introduction.

Theorem 9. *Let $(M, \langle\, , \rangle)$ be a complete, m-dimensional Riemannian manifold with a pole o and radial sectional curvature K_{rad} with respect to o satisfying*

$$K_{\mathrm{rad}}(x) \le -G\big(r(x)\big), \tag{159}$$

for some $G \in C^0(\mathbb{R}^+_0)$. Let $g \in C^2(\mathbb{R}^+_0)$ be a solution of

$$\begin{cases} g'' - Gg \le 0 & \text{on } \mathbb{R}^+, \\ g(0) = 0, & g'(0) = 1. \end{cases} \tag{160}$$

Suppose that $g > 0$ on $\mathbb{R}^+$ and that $v = g^{m-1}$ satisfies (V_{L1}), and set $\chi = \chi_v$ as usual. Let $q(x), b(x) \in \mathrm{H\ddot{o}l}_{\mathrm{loc}}(M)$ be such that

$$|q(x)| \le A(r(x)), \qquad |b(x)| \le B(r(x)), \tag{161}$$

for some non-negative $A, B \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+_0)$ with

$$A(r) \le \chi(r) \quad \text{on } \mathbb{R}^+, \qquad \frac{A(r)}{\sqrt{\chi(r)}} \in L^1(+\infty), \qquad \frac{B(r)}{\sqrt{\chi(r)}} \in L^1(+\infty). \tag{162}$$

Fix $\sigma > 1$. Then, there exists a constant $\beta > 0$, depending on σ, g, A, B such that, for each $\gamma_\infty \in (0, \beta)$, there exist $0 < \Gamma_1 \le \Gamma_2$ and a solution $u \in \mathrm{H\ddot{o}l}^2_{\mathrm{loc}}(M)$ (C^∞ if q, b are C^∞) of

$$\Delta u + q(x)u - b(x)u^\sigma = 0 \tag{163}$$

satisfying

$$\Gamma_1 \le u(x) \le \Gamma_2 \qquad \text{on } M \tag{164}$$

and

$$\lim_{r(x) \to +\infty} u(x) = \gamma_\infty. \tag{165}$$

Moreover, $\Gamma_2 \to 0$ as $\gamma_\infty \to 0$.

Proof. Consider the solutions h_j of the Cauchy problems

$$\begin{cases} (vh_1')' - Avh_1 = 0 & \text{on } \mathbb{R}^+ \\ h_1(0) = 1, \quad h_1'(0) = 0 \end{cases}, \qquad \begin{cases} (vh_2')' + Avh_2 = 0 & \text{on } \mathbb{R}^+ \\ h_2(0) = 1, \quad h_2'(0) = 0. \end{cases} \tag{166}$$

Note that, since $A \geq 0$ and $A \leq \chi$, $h_j > 0$ on $\mathbb{R}^+$ and a first integration shows that, respectively,

$$h_1' \geq 0, \qquad h_2' \leq 0. \tag{167}$$

This will be essential in the construction of the sub- and supersolution. By (162) and Theorem 5, $h_j(r) \to C_j$ as $r \to +\infty$, for some $C_j > 0$. Sturm arguments give that $h_1 \geq h_2$, and so $C_1 \geq C_2$. From Theorem 7, for each j there exists a positive β_j, depending on σ, g, A, B, such that, for each fixed $\gamma_{\infty,j} \in (0, \beta_j)$, we can find $0 < \gamma_{0,j} < \gamma_{M,j}$ and a positive solution γ_j of, respectively,

$$\begin{cases} (v\gamma_1')' - Av\gamma_1 = Bv\gamma_1^\sigma & \text{on } \mathbb{R}^+ \\ \gamma_1(0) = \gamma_{0,1}, \quad \gamma_1'(0) = 0 \end{cases}, \qquad \begin{cases} (v\gamma_2')' + Av\gamma_2 = -Bv\gamma_2^\sigma & \text{on } \mathbb{R}^+ \\ \gamma_2(0) = \gamma_{0,2}, \quad \gamma_2'(0) = 0 \end{cases} \tag{168}$$

with the properties

$$\gamma_j(r) \leq \gamma_{M,j} h_j(r) \quad \text{on } \mathbb{R}^+, \qquad \gamma_j(r) \to \gamma_{\infty,j} C_j \qquad \text{as } r \to +\infty. \tag{169}$$

Set

$$\bar{\beta} = \min\{\beta_1, \beta_2\}, \qquad \beta = \min\{C_2, 1\}\bar{\beta}.$$

We restrict our choice of γ_∞ to $(0, \beta)$. Now, fix $\gamma_{\infty,2}$ in such a way that

$$\gamma_{\infty,2} C_2 = \gamma_\infty, \tag{170}$$

and note that $\gamma_{\infty,2} = \gamma_\infty / C_2 \in (0, \beta/C_2) \subseteq (0, \bar{\beta})$. Combining (170) with the property that $\gamma_{M,2} \to 0$ as $\gamma_{\infty,2} \to 0$ granted by Theorem 7, we deduce

$$\gamma_{M,2} \to 0 \qquad \text{as} \qquad \gamma_\infty \to 0. \tag{171}$$

Next, we let

$$\gamma_{\infty,1} = \gamma_{\infty,2} \frac{C_2}{C_1} \leq \gamma_{\infty,2}.$$

In this way, $\gamma_{\infty,1} \in (0, \bar{\beta})$ and, by (169),

$$\frac{\gamma_1(r)}{\gamma_2(r)} \to 1 \qquad \text{as } r \to +\infty. \tag{172}$$

Define $\xi_j = \gamma_j / h_j$. Then, ξ_j satisfies

$$\begin{cases} (h_1^2 v \xi_1')' = (Bh_1^{\sigma-1}) h_1^2 v \xi_1^\sigma & \text{on } \mathbb{R}^+ \\ \xi_1(0) = \gamma_{0,1}, \quad \xi_1'(0) = 0 \end{cases} \qquad \begin{cases} (h_2^2 v \xi_2')' = -(Bh_2^{\sigma-1}) h_2^2 v \xi_2^\sigma & \text{on } \mathbb{R}^+ \\ \xi_2(0) = \gamma_{0,2}, \quad \xi_2'(0) = 0. \end{cases} \tag{173}$$

A first integration shows that $\xi_1' \geq 0$ and $\xi_2' \leq 0$. Fix $x \in M$ and let c be a ray joining o to x. Having defined w_c as in (156), assumption (159) and the Laplacian comparison theorem yield $w_c'/w_c \geq v'/v$ on $\mathbb{R}^+$, and with a simple computation we get

$$\frac{(h_j^2 v)'}{h_j^2 v} \leq \frac{(h_j^2 w_c)'}{h_j^2 w_c} \qquad \text{for each } j.$$

Consequently, by (173)

$$\begin{aligned} 0 &= \xi_1'' + \frac{(h_1^2 v)'}{h_1^2 v}\xi_1' - (Bh_1^{\sigma-1})\xi_1^\sigma \le \xi_1'' + \frac{(h_1^2 w_c)'}{h_1^2 w_c}\xi_1' - (Bh_1^{\sigma-1})\xi_1^\sigma; \\ 0 &= \xi_2'' + \frac{(h_2^2 v)'}{h_2^2 v}\xi_2' + (Bh_2^{\sigma-1})\xi_2^\sigma \ge \xi_2'' + \frac{(h_2^2 w_c)'}{h_2^2 w_c}\xi_2' + (Bh_2^{\sigma-1})\xi_2^\sigma. \end{aligned} \tag{174}$$

Whence, a computation that uses the first of (166) and (173) shows that γ_1 solves, on $\mathbb{R}^+$,

$$\gamma_1'' + \frac{w_c'}{w_c}\gamma_1' - A\gamma_1 - B\gamma_1^\sigma \ge \xi_1 h_1' \left(\frac{w_c'}{w_c} - \frac{v'}{v}\right). \tag{175}$$

Again by the Laplacian comparison theorem and since $h_1' \ge 0$, the right hand side is non-negative, which implies that γ_1 is a solution of

$$\begin{cases} (w_c\gamma_1')' - Aw_c\gamma_1 \ge Bw_c\gamma_1^\sigma \\ \gamma_1(0) = \gamma_{0,1}, \quad \gamma_1'(0) = 0. \end{cases} \tag{176}$$

Note the additional property

$$\gamma_1' = \xi_1' h_1 + h_1'\xi_1 \ge 0 \qquad \text{on } \mathbb{R}^+. \tag{177}$$

Setting

$$u_1(x) = \gamma_1(r(x)),$$

we have, by (158) and (176),

$$\begin{aligned} \Delta u_1 + q(x)u_1 - b(x)u_1^\sigma &\ge \Delta u_1 - A(r)u_1 - B(r)u_1^\sigma \\ &= \gamma_1'' + \gamma_1'\frac{w_c'}{w_c} - A\gamma_1 - B\gamma_1^\sigma \ge 0, \end{aligned}$$

showing that u_1 is a subsolution of (163). Similarly, using the second of (166) and (173), γ_2 solves

$$\gamma_2'' + \frac{w_c'}{w_c}\gamma_2' + A\gamma_2 + B\gamma_2^\sigma \le \xi_2 h_2' \left(\frac{w_c'}{w_c} - \frac{v'}{v}\right). \tag{178}$$

Now, since $h_2' \le 0$, applying again the Laplacian comparison theorem we obtain that the right hand side is non-positive, and also

$$\gamma_2' = \xi_2' h_2 + h_2'\xi_2 \le 0 \qquad \text{on } \mathbb{R}^+. \tag{179}$$

By the same arguments as above, the function $u_2(x) = \gamma_2(r(x))$ is a supersolution of (163). Combining the two monotonicity properties (177), (179) with the asymptotic behaviour (172) we deduce that $\gamma_1 \le \gamma_2$ on $\mathbb{R}^+$, so

$$u_1(x) \le u_2(x) \qquad \text{on } M.$$

The monotone iteration scheme yields the existence of $u \in \mathrm{H\ddot{o}l}^2_{\mathrm{loc}}(M_g)$ solving (163) and such that $u_1 \le u \le u_2$. From

$$\begin{aligned} &u(x) \le \gamma_2(r(x)) \le \gamma_{M,2}h_2(r(x)) \le \gamma_{M,2} \qquad (\text{since } h_2' \le 0,\ h_2(0) = 1); \\ &u_j(x) = \gamma_j(r(x)) \to \gamma_\infty \qquad \text{as } r(x) \to +\infty, \end{aligned}$$

property (164) follows at once by setting $\Gamma_2 = \gamma_{M,2}$, and the validity of (171) concludes the proof. Note that the C^∞-smoothness of u, when $q, b \in C^\infty(M)$, follows from elliptic regularity. $\square$

Proof of Corollary 2. We set $q(x) = 0$, and choose $A = 0$, $k = 0$ in Theorem 9, so that the first two conditions in (162) are trivially satisfied. By computations (254), (257) and (258) in the Appendix, for $\alpha \geq -2$

$$\chi(r) \sim Cr^{\alpha} \qquad \text{as } r \to +\infty,$$

for some explicit constant $C > 0$, thus assumption (44) on B is exactly the third of (162), and the conclusion follows from a plain application of the theorem. □

The case $k \in (0, 1)$.

In this section, we prove Corollary 1 and Theorem 2 of the Introduction, as consequences of the general Theorem 10 below. In this latter, we restrict to the case $q(x) \geq 0$, and we consider non-negative A_1, A_2 such that

$$A_1\big(r(x)\big) \leq q(x) \leq A_2\big(r(x)\big).$$

The strategy of proof is close, in spirit, to that of Theorem 9, and relies on the construction of suitable h_j, γ_j, ξ_j. The supersolution $u_+(x)$ is basically obtained in the same way as in the case $k = 0$, and only requires the upper bound A_2 and the geometric assumption

$$K_{\mathrm{rad}}(x) \leq -G\big(r(x)\big). \tag{180}$$

However, in order to produce the subsolution, one has to take into account the monotonicity of the solution of the linear ODE coming from A_1, which, unfortunately, is opposite to the corresponding one in Theorem 9. This makes necessary the use of the Laplacian comparison theorem from above under the lower bound

$$K_{\mathrm{rad}}(x) \geq -\bar{G}\big(r(x)\big). \tag{181}$$

However, in order to match the correct inequalities we shall carefully mix solutions of ODEs depending on the "upper" volume $\bar{v}$ coming from (181) (the function $\bar{h}$ below) with that depending on the "lower" volume v constructed from (180) (the function ξ_1). The drawback of this method is that the subsolution $u_-(x)$ is likely to be above the supersolution, in the sense that

$$u_+(x) = o\Big(u_-(x)\Big) \qquad \text{as } r(x) \to +\infty.$$

Nevertheless, if G and $\bar{G}$ are sufficiently close (see (189) below), we are able to avoid this basic problem. As it will become apparent in the proof, we achieve our goal via the next two lemmas.

Lemma 3. *Let M_g, $M_{\bar{g}}$ be models satisfying, respectively, $v^{-1} = g^{1-m} \in L^1(+\infty)$ and $\bar{v}^{-1} = \bar{g}^{1-m} \in L^1(+\infty)$. Let χ_v, $\chi_{\bar{v}}$ be the associated critical curves. Suppose that*

$$C^{-1}v(r) \leq \bar{v}(r) \leq Cv(r) \qquad \text{on } \mathbb{R}^+, \tag{182}$$

for some $C > 0$. Let $\bar{H}_k$ be as in (58) with v substituted by $\bar{v}$. Then, if $k < 1$,

$$\frac{\chi_{\bar{v}}(r)}{\sqrt{\chi_{\bar{H}_k^2 \bar{v}}(r)}} - \frac{\chi_v(r)}{\sqrt{\chi_{H_k^2 v}(r)}} \in L^1(+\infty). \tag{183}$$

Furthermore, suppose that $\bar v/v$ is non-decreasing. Then,

$$0 \le \chi_v(r)\left(\frac{1}{\sqrt{\chi_{H_k^2 v}(r)}} - \frac{1}{\sqrt{\chi_{\bar H_k^2 \bar v}(r)}}\right) \in L^1(+\infty). \tag{184}$$

Proof. We begin proving (183). From the first one in (59) we deduce

$$\frac{\chi_{\bar v}(r)}{\sqrt{\chi_{\bar H_k^2 \bar v}(r)}} - \frac{\chi_v(r)}{\sqrt{\chi_{H_k^2 v}(r)}} = \frac{1}{\sqrt{1-k}}\Big(\sqrt{\chi_{\bar v}(r)} - \sqrt{\chi_v(r)}\Big). \tag{185}$$

By the very definition of the critical curves χ_v, $\chi_{\bar v}$, for $0 < R < r$

$$\int_R^r \Big(\sqrt{\chi_{\bar v}(s)} - \sqrt{\chi_v(s)}\Big)\mathrm{d}s = -\frac{1}{2}\log\left(\int_r^{+\infty}\frac{\mathrm{d}s}{\bar v(s)}\Big/\int_r^{+\infty}\frac{\mathrm{d}s}{v(s)}\right) + C(R),$$

for a constant $C(R) \in \mathbb{R}$. Letting $r \to +\infty$ and using (182) we conclude that $\sqrt{\chi_{\bar v}(r)} - \sqrt{\chi_v(r)} \in L^1(+\infty)$, from which (183) follows.
To prove (184), we first observe that the assumption that $\bar v/v$ be non-decreasing implies, via Proposition 3.12 in [1], that $\chi_{\bar v} \ge \chi_v$ on $\mathbb{R}^+$. Therefore, using (59) we infer

$$\chi_{\bar H_k^2 \bar v}(r) = (1-k)\chi_{\bar v}(r) \ge (1-k)\chi_v(r) = \chi_{H_k^2 v}(r),$$

thus

$$\begin{aligned}
0 \ &\le \ \chi_v(r)\left(\frac{1}{\sqrt{\chi_{H_k^2 v}(r)}} - \frac{1}{\sqrt{\chi_{\bar H_k^2 \bar v}(r)}}\right) = \chi_v(r)\frac{\sqrt{\chi_{\bar H_k^2 \bar v}(r)} - \sqrt{\chi_{H_k^2 v}(r)}}{\sqrt{\chi_{H_k^2 v}(r)}\sqrt{\chi_{\bar H_k^2 \bar v}(r)}} \\
&\le \ \chi_v(r)\frac{\sqrt{\chi_{\bar H_k^2 \bar v}(r)} - \sqrt{\chi_{H_k^2 v}(r)}}{\chi_{H_k^2 v}(r)} = \frac{1}{1-k}\sqrt{\chi_{\bar H_k^2 \bar v}(r)} - \frac{1}{\sqrt{1-k}}\sqrt{\chi_v(r)} \\
&= \ \frac{1}{\sqrt{1-k}}\Big(\sqrt{\chi_{\bar v}(r)} - \sqrt{\chi_v(r)}\Big),
\end{aligned}$$

and (184) follows from (185) and (183). □

Remark 25. The restriction $k < 1$ in the above result is substantial. Indeed, if $k = 1$, by the second equation of (59) it holds

$$\begin{aligned}
&\int_R^r \left(\frac{\chi_{\bar v}(s)}{\sqrt{\chi_{\bar H_k^2 \bar v}(s)}} - \frac{\chi_v(s)}{\sqrt{\chi_{H_k^2 v}(s)}}\right)\mathrm{d}s = +\frac{1}{4}\left[\log^2\int_r^{+\infty}\frac{\mathrm{d}s}{\bar v(s)} - \log^2\int_r^{+\infty}\frac{\mathrm{d}s}{v(s)}\right] + C(R) \\
&= \frac{1}{4}\left[\log\left(\int_r^{+\infty}\frac{\mathrm{d}s}{\bar v(s)}\int_r^{+\infty}\frac{\mathrm{d}s}{v(s)}\right)\right]\left[\log\left(\int_r^{+\infty}\frac{\mathrm{d}s}{\bar v(s)}\Big/\int_r^{+\infty}\frac{\mathrm{d}s}{v(s)}\right)\right] + C(R).
\end{aligned}$$

Now, suppose that $\bar v(r) \sim Cv(r)$ as r diverges, for some $C \ne 1$. Then, the RHS diverges as $r \to +\infty$.

Lemma 4. *In the assumptions of Lemma 3, let $A \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+_0)$ be such that*

$$0 \le A(r) \le \chi_v(r).$$

Then, if $k<1$,

$$\frac{A(r)-k\chi_{\bar v}(r)}{\sqrt{\chi_{\bar H_k^2\bar v}(r)}}\in L^1(+\infty)\qquad \textit{if and only if}\qquad \frac{A(r)-k\chi_v(r)}{\sqrt{\chi_{H_k^2 v}(r)}}\in L^1(+\infty). \tag{186}$$

Proof. First, observe that

$$0\le A(r)\left(\frac{1}{\sqrt{\chi_{H_k^2 v}(r)}}-\frac{1}{\sqrt{\chi_{\bar H_k^2\bar v}(r)}}\right)\le \chi_v(r)\left(\frac{1}{\sqrt{\chi_{H_k^2 v}(r)}}-\frac{1}{\sqrt{\chi_{\bar H_k^2\bar v}(r)}}\right).$$

Then, apply Lemma 3 to the identity

$$\frac{A-k\chi_{\bar v}}{\sqrt{\chi_{\bar H_k^2\bar v}}}=\frac{A-k\chi_v}{\sqrt{\chi_{H_k^2 v}}}-A\left(\frac{1}{\sqrt{\chi_{H_k^2 v}}}-\frac{1}{\sqrt{\chi_{\bar H_k^2\bar v}}}\right)-k\left(\frac{\chi_{\bar v}}{\sqrt{\chi_{\bar H_k^2\bar v}}}-\frac{\chi_v}{\sqrt{\chi_{H_k^2 v}}}\right).$$

□

We are now ready to prove

Theorem 10. *Let* $(M,\langle\,,\,\rangle)$ *be a complete,* m*-dimensional Riemannian manifold with a pole* o *and radial sectional curvature* K_{rad} *with respect to* o *satisfying*

$$-\bar G\big(r(x)\big)\le K_{\mathrm{rad}}(x)\le -G\big(r(x)\big), \tag{187}$$

for some $G,\bar G\in C^0(\mathbb{R}_0^+)$*. Let* $g\in C^2(\mathbb{R}_0^+)$ *be the solution of*

$$\begin{cases} g''-Gg=0 & \text{on } \mathbb{R}^+,\\ g(0)=0, & g'(0)=1.\end{cases} \tag{188}$$

Suppose that $g>0$ *on* $\mathbb{R}^+$ *and that both*

$$g^{-2}\in L^1(+\infty),\qquad \text{and}\qquad v^{-1}=g^{1-m}\in L^1(+\infty).$$

Set $\chi=\chi_v$ *as usual. Assume that* $\bar G$ *is close enough to* G *in the following sense:*

$$\frac{\bar G(r)-G(r)}{\sqrt{\chi_{g^2}(r)}}\in L^1(+\infty). \tag{189}$$

Let $q(x)\in \mathrm{H\ddot{o}l}_{\mathrm{loc}}(M)$ *be such that*

$$0\le A_1\big(r(x)\big)\le q(x)\le A_2\big(r(x)\big),$$

for some $A_j\in L^\infty_{\mathrm{loc}}(\mathbb{R}_0^+)$*,* $j\in\{1,2\}$ *such that* $A_2(r)\le\chi(r)$*. Suppose that there exists* $k\in(0,1)$ *such that*

$$\frac{A_j(r)-k\chi(r)}{\sqrt{\chi(r)}}\in L^1(+\infty)\qquad \text{for } j=1,2. \tag{190}$$

Let $\sigma>1$*,* $B\in L^\infty_{\mathrm{loc}}(\mathbb{R}_0^+)$ *and assume*

$$\frac{B(r)H_k(r)^{\sigma-1}}{\sqrt{\chi(r)}}\in L^1(+\infty). \tag{191}$$

Then, for each $b(x) \in \mathrm{H\ddot{o}l}_{\mathrm{loc}}(M)$ satisfying

$$|b(x)| \le B(r(x)) \qquad \text{on } M,$$

there exist $0 < \Gamma_1 \le \Gamma_2$ such that the equation

$$\Delta u + q(x)u - b(x)u^\sigma = 0 \tag{192}$$

possesses a positive, bounded solution $u \in \mathrm{H\ddot{o}l}^2_{\mathrm{loc}}(M)$ (C^∞ if q, b are C^∞) with the property that, for some $r_0 > 0$,

$$\Gamma_1 H_k(r(x)) \le u(x) \le \Gamma_2 H_k(r(x)) \qquad \text{for } r(x) \ge r_0. \tag{193}$$

Moreover, $\|u\|_{L^\infty(M)}$ and Γ_2 can be chosen to be as small as we wish.

Remark 26. Due to the presence of the factor $H_k(r)^{\sigma-1}$ and since $k \in (0,1)$, condition (191) allows ample oscillations of $b(x)$ between positive and negative values. For the two relevant classes of $G(r)$ described in (38), the reader can easily check this claim with the aid of the computations in the Appendix.

Remark 27. By Remark 14, the requirement (190) gives a family of independent conditions as k varies. In particular, since $k > 0$ neither of them implies

$$\frac{A(r)}{\sqrt{\chi(r)}} \in L^1(+\infty),$$

thus Theorem 10 is not contained in Theorem 9.

Proof of Theorem 10. We begin with the construction of the supersolution, and let h solve

$$\begin{cases} (vh')' + A_2 v h = 0 & \text{on } \mathbb{R}^+ \\ h(0) = 1, \qquad h'(0) = 0. \end{cases} \tag{194}$$

Since $0 \le A_2 \le \chi$, h is positive and non-increasing, so $h \le 1$ on $\mathbb{R}^+$. Moreover, since $k < 1$, $\chi_{H_k^2 v}(r) = (1-k)\chi(r)$ and so (190) implies (85) with $A = A_2$. An application of Theorem 5 shows that

$$h(r) \sim C_2 H_k(r) \qquad \text{as } r \to +\infty, \tag{195}$$

for some $C_2 > 0$. By Theorem 7, for each $\gamma_{\infty,2}$ sufficiently small, there exists a positive solution γ_2 of

$$\begin{cases} (v\gamma_2')' + A_2 v \gamma_2 = -Bv\gamma_2^\sigma & \text{on } \mathbb{R}^+ \\ \gamma_2(0) = \gamma_{0,2}, \quad \gamma_2'(0) = 0 \end{cases} \tag{196}$$

with the further properties

$$\gamma_2(r) \le \gamma_{M,2} h(r) \le \gamma_{M,2} \quad \text{on } \mathbb{R}^+, \qquad \frac{\gamma_2(r)}{h(r)} \to \gamma_{\infty,2} \qquad \text{as } r \to +\infty \tag{197}$$

for some $0 < \gamma_{0,2} \le \gamma_{M,2}$ satisfying $\gamma_{M,2} \to 0$ as $\gamma_{\infty,2} \to 0$. Indeed, by (195) and since $k < 1$, (133) is equivalent to (191). Having set c, w_c as in (156), by (157) and the Laplacian comparison theorem, condition $K_{\mathrm{rad}} \le -G(r)$ in (187) implies

$$\Delta r\big(c(s)\big) = \frac{w_c'}{w_c}(s) \ge \frac{v'}{v}(s) \qquad \text{on } \mathbb{R}^+, \tag{198}$$

therefore

$$\frac{(h^2 w_c)'}{h^2 w_c} \geq \frac{(h^2 v)'}{h^2 v}. \tag{199}$$

We set $\xi_2 = \gamma_2/h$ and we observe that ξ_2 solves

$$\begin{cases} \xi_2'' + \dfrac{(h^2 v)'}{h^2 v}\xi_2' = -(Bh^{\sigma-1})\xi_2^\sigma \\ \xi_2(0) = \gamma_{0,2}, \qquad \xi_2'(0) = 0, \end{cases} \tag{200}$$

in particular $\xi_2' \leq 0$. Therefore, from (199) it follows that

$$\xi_2'' + \frac{(h^2 w_c)'}{h^2 w_c}\xi_2' \leq -(Bh^{\sigma-1})\xi_2^\sigma.$$

The same computation performed in Theorem 9 shows that $\gamma_2 = h\xi_2$ solves

$$\gamma_2'' + \frac{w_c'}{w_c}\gamma_2' + A_2\gamma_2 + B\gamma_2^\sigma \leq \xi_2 h' \left(\frac{w_c'}{w_c} - \frac{v'}{v} \right). \tag{201}$$

Combing $h' \leq 0$ and (198) we deduce that γ_2 is a solution of

$$\begin{cases} (w_c\gamma_2')' + A_2 w_c \gamma_2 \leq -Bw_c\gamma_2^\sigma \qquad \text{on } \mathbb{R}^+ \\ \gamma_2(0) = \gamma_{0,2}, \quad \gamma_2'(0) = 0. \end{cases} \tag{202}$$

From $q(x) \leq A_2(r(x))$ and $-b(x) \leq B(r(x))$ we conclude that $u_+(x) = \gamma_2(r(x))$ is a supersolution of (192) satisfying (by (195) and (197))

$$u_+(x) \leq \gamma_{M,2} \quad \text{on } \mathbb{R}^+, \qquad u_+(x) \sim \gamma_{\infty,2} C_2 H_k(r(x)) \qquad \text{as } r(x) \to +\infty. \tag{203}$$

To construct the subsolution, let $\bar{g}$ solve

$$\begin{cases} \bar{g}'' - \bar{G}\bar{g} = 0 \qquad \text{on } \mathbb{R}^+, \\ \bar{g}(0) = 0, \qquad \bar{g}'(0) = 1. \end{cases}$$

and set $\bar{v} = \bar{g}^{m-1}$. Since $\bar{G} \geq G$, by Sturm arguments $\bar{g}/g$ is non-decreasing, and thus also $\bar{v}/v$. Hence, $\chi_{\bar{v}}$ is well defined and $\chi \leq \chi_{\bar{v}}$ by Proposition 3.12 in [1]. Since, by assumption, $A_1 \leq A_2 \leq \chi$, it follows that $A_1 \leq \chi_{\bar{v}}$, and thus the problem

$$\begin{cases} (\bar{v}\bar{h}')' + A_1\bar{v}\bar{h} = 0 \qquad \text{on } \mathbb{R}^+ \\ \bar{h}(0) = 1, \qquad \bar{h}'(0) = 0 \end{cases} \tag{204}$$

has a positive, non-increasing solution $\bar{h}$, thus satisfying $\bar{h} \leq 1$ on $\mathbb{R}^+$. By (189) and the fact that $\bar{G} \geq G$, both conditions in (95) are met, and

$$\bar{v}(r) \sim Cv(r) \qquad \text{as } r \to +\infty, \tag{205}$$

for some positive constant C. We next determine the asymptotic behaviour of $\bar{h}$. By Lemma 4 with $A = A_1$, and again since $\chi_{H_k^2 v}(r) = (1-k)\chi(r)$, our requirement (190) is equivalent to

$$\frac{A_1(r) - k\chi_{\bar{v}}(r)}{\sqrt{\chi_{\bar{H}_k^2\bar{v}}(r)}} \in L^1(+\infty).$$

An application of Theorem 5 yields $\bar{h}(r) \sim C\bar{H}_k(r)$ as $r \to +\infty$. Finally, since $\bar{v} \sim Cv$ as r diverges, we get

$$\bar{h}(r) \sim C_1 H_k(r) \qquad \text{as } r \to +\infty, \tag{206}$$

for some positive C_1. As a consequence, assumption (191) is equivalent to

$$\frac{B(r)\bar{h}(r)^{\sigma-1}}{\sqrt{\chi_{\bar{h}^2 v}(r)}} \in L^1(+\infty). \tag{207}$$

Now, we apply the existence Theorem 7 with the choices $k = 0$, $A = 0$, B (in Theorem 7) equals to $B\bar{h}^{\sigma-1}$ and v (in Theorem 7) given by $\bar{h}^2 v$. Note that, with our choices, the solution of (132) is the constant 1. Thus, for each positive $\gamma_{\infty,1}$ sufficiently small there exists a positive solution ξ_1 of

$$\begin{cases} (\bar{h}^2 v \xi_1')' = (B\bar{h}^{\sigma-1})\bar{h}^2 v \xi_1^\sigma \\ \xi_1(0) = \gamma_{0,1}, \qquad \xi_1'(0) = 0, \end{cases} \tag{208}$$

with the properties that

$$\xi_1(r) \le \gamma_{M,1} \quad \text{on } \mathbb{R}^+, \qquad \xi(r) \to \gamma_{\infty,1} \quad \text{as } r \to +\infty, \tag{209}$$

for some positive $\gamma_{0,1} \le \gamma_{M,1}$ satisfying $\gamma_{M,1} \to 0$ as $\gamma_{\infty,1} \to 0$. Integrating (208) once, we get $\xi_1' \ge 0$. It is important to observe that, although the solution $\bar{h}$ of the linear ODE (204) is related to the volume $\bar{v}$ of the model $M_{\bar{g}}$, the function ξ solves an ODE whose weighted volume $\bar{h}^2 v$ depends on the volume v of the model M_g.
We first observe that, using (198)

$$\frac{(\bar{h}^2 w_c)'}{\bar{h}^2 w_c} \ge \frac{(\bar{h}^2 v)'}{\bar{h}^2 v},$$

whence coupling with $\xi_1' \ge 0$ we get

$$\xi_1'' + \frac{(\bar{h}^2 w_c)'}{\bar{h}^2 w_c}\xi_1' \ge \xi_1'' + \frac{(\bar{h}^2 v)'}{\bar{h}^2 v}\xi_1' = (Bh^{\sigma-1})\xi_1^\sigma. \tag{210}$$

Define $\gamma_1 = \bar{h}\xi_1$. Then, from a combination of (206) and (209) and since $\bar{h} \le 1$ we infer the relations

$$\gamma_1(r) \le \gamma_{M,1}\bar{h}(r) \le \gamma_{M,1} \quad \text{on } \mathbb{R}^+, \qquad \gamma_1(r) \sim \gamma_{\infty,1} C_1 H_k(r) \qquad \text{as } r \to +\infty. \tag{211}$$

A computation using (210) and (204) shows that

$$\gamma_1'' + \frac{w_c'}{w_c}\gamma_1' + A_1\gamma_1 - B\gamma_1^\sigma \ge \xi_1 \bar{h}' \left(\frac{w_c'}{w_c} - \frac{\bar{v}'}{\bar{v}} \right). \tag{212}$$

The Laplacian comparison theorem from above and assumption $K_{\mathrm{rad}} \ge -\bar{G}(r)$ imply the inequality

$$\Delta r\big(c(s)\big) = \frac{w_c'}{w_c}(s) \le \frac{\bar{v}'}{\bar{v}}(s), \tag{213}$$

thus putting together with $\bar{h}' \le 0$ we deduce from (212) that γ_1 solves

$$\begin{cases} (w_c \gamma_1')' + A_1 w_c \gamma_1 \ge B w_c \gamma_1^\sigma \qquad \text{on } \mathbb{R}^+ \\ \gamma_1(0) = \gamma_{0,1}, \quad \gamma_1'(0) = 0. \end{cases} \tag{214}$$

Using $q(x) \geq A_1(r(x))$ and $b(x) \leq B(r(x))$ we conclude that $u_-(x) = \gamma_1(r(x))$ is a subsolution of (192) satisfying (by (206) and (211))

$$u_-(x) \leq \gamma_{M,1} \quad \text{on } \mathbb{R}^+, \qquad u_-(x) \sim \gamma_{\infty,1} C_1 H_k(r(x)) \qquad \text{as } r(x) \to +\infty. \tag{215}$$

Now, by inspecting the relations (203) and (215), and using that $\gamma_{M,1} \to 0$ as $\gamma_{\infty,1} \to 0$, for each fixed $\gamma_{\infty,2}$ we can choose $\gamma_{\infty,1}$ small enough that $u_- \leq u_+$ on M. Now, applying the monotone iteration scheme gives the desired solution $u \in \mathrm{Höl}^2_{\mathrm{loc}}(M)$ satisfying $u_- \leq u \leq u_+$. From $\|u_+\|_{L^\infty(M)} = \gamma_{M,2}$, up to choosing $\gamma_{\infty,2}$ appropriately we can make $\|u\|_{L^\infty(M)}$ as small as we wish. This concludes the proof. The C^∞-regularity of u, when $q, b \in C^\infty(M)$, follows from elliptic regularity theory. □

Next, we prove Corollary 1 in the Introduction.

Proof of Corollary 1. Set $G(r) = 0$, $g(r) = r$, and note that, since $m \geq 3$, $v = g^{m-1}$ satisfies (V_{L1}). Taking into account that

$$\chi(r) = \frac{(m-2)^2}{4r^2},$$

and, by (260) and (261) in the Appendix, for $k \in (0,1)$

$$\begin{aligned} H_k(r) \quad &\sim \quad C r^{-\frac{m-2}{2}(1-\sqrt{1-k})} \\ \chi_{H_k^2 v}(r) \quad &\sim \quad \frac{(m-2)^2(1-k)}{4r^2} \end{aligned}$$

as $r \to +\infty$, the result follows immediately from Theorem 10. □

Finally, specializing to hyperbolic type settings, we obtain Theorem 2.

Proof of Theorem 2. The conformal factor u in (2) has to satisfy

$$\Delta u - \frac{s(x)}{c_m} u + \frac{\widetilde{s}(x)}{c_m} u^\sigma = 0, \qquad c_m = \frac{4(m-1)}{m-2}, \quad \sigma = \frac{m+2}{m-2}. \tag{216}$$

We set $G(r) = -H^2$ and $g(r) = H^{-1}\sinh(Hr)$ in Theorem 10, so that we compare M with the hyperbolic space $\mathbb{H}^m_H$. Clearly, $v(r) = g(r)^{m-1}$ satisfies (V_{L1}), and by (56) it holds

$$\chi(r) > \frac{H^2(m-1)^2}{4} \quad \text{on } \mathbb{R}^+, \qquad \chi(r) \to \frac{H^2(m-1)^2}{4} \quad \text{as } r \to +\infty. \tag{217}$$

We define

$$k = \frac{m(m-2)}{(m-1)^2} < 1, \qquad q(x) = -\frac{s(x)}{c_m}, \qquad b(x) = -\frac{\widetilde{s}(x)}{c_m}. \tag{218}$$

Tracing relation (18) we deduce

$$-m(m-1)H^2 - m(m-1)\mathcal{K}(r) \leq s(x) \leq -m(m-1)H^2.$$

From (20) and (217) we thus get

$$\begin{cases} q(x) \leq \dfrac{1}{c_m} \dfrac{(m-1)^3 H^2}{m-2} = \dfrac{(m-1)^2 H^2}{4} \leq \chi(r), \\[2ex] \dfrac{m(m-2)H^2}{4} \leq q(x) \leq \dfrac{m(m-2)H^2}{4} + \bar{\mathcal{K}}(r). \end{cases} \tag{219}$$

Where

$$0 \leq \bar{\mathcal{K}}(r) = \frac{m(m-1)}{c_m}\mathcal{K}(r) \in L^1(+\infty).$$

Choose

$$A_1(r) = \frac{m(m-2)H^2}{4}, \qquad A_2(r) = \min\left\{\frac{m(m-2)H^2}{4} + \bar{\mathcal{K}}(r), \frac{(m-1)^2H^2}{4}\right\}.$$

and observe that

$$A_1\big(r(x)\big) \leq q(x) \leq A_2\big(r(x)\big) \leq \chi\big(r(x)\big)$$

and that A_2 can be written as

$$A_2(r) = \frac{m(m-2)H^2}{4} + \hat{\mathcal{K}}(r), \qquad \text{for some } 0 \leq \hat{\mathcal{K}}(r) \in L^1(+\infty).$$

We recall the next asymptotic expansion (see (57)):

$$\chi(r) = \frac{1}{4}\left\{\frac{1}{(m-1)H} + \frac{m-1}{(m+1)H}e^{-2Hr} + o\left(e^{-2Hr}\right)\right\}^{-2} \qquad \text{as } r \to +\infty.$$

Keeping in mind our definition of k in (218), we deduce

$$\begin{aligned} A_1(r) - k\chi(r) &= -\frac{1}{2}\frac{m(m-1)^2(m-2)}{m+1}H^2e^{-2Hr} + o\left(e^{-2Hr}\right) \\ A_2(r) - k\chi(r) &= -\frac{1}{2}\frac{m(m-1)^2(m-2)}{m+1}H^2e^{-2Hr} + o\left(e^{-2Hr}\right) + \hat{\mathcal{K}}(r). \end{aligned} \tag{220}$$

Now, again by the choice of k (see computation (256) in the Appendix),

$$H_k(r) \sim Ce^{-\frac{H(m-1)}{2}(1-\sqrt{1-k})r} = Ce^{-\frac{H(m-2)}{2}r} \qquad \text{as } r \to +\infty. \tag{221}$$

Combining (220) and (221), the integrability condition (190) follows because of that of $\hat{\mathcal{K}}(r)$. On the other hand, by (221) and our definitions of σ and k, requirement (191) becomes

$$e^{-2Hr}B(r) \in L^1(+\infty).$$

Applying Theorem 10 we reach the desired conclusion. □

The case $k \in (-\infty, 1]$ on models.

When M itself is radially symmetric, the situation gets simpler. Indeed, in this case $\bar{G} = G$, and there is no need to match the monotonicity of h, $\bar{h}$ with the inequalities deriving from the Laplacian comparison theorems. Moreover, as $G = \bar{G}$ the restriction $k < 1$ made in order to use Lemma 4 (see also Remark 25) is not necessary anymore. With a procedure close to the one presented in Theorems 9 and 10, one can easily prove the next result:

Theorem 11. *Let M_g be model manifold such that $v = g^{m-1}$ satisfies (V_{L1}), and set $\chi = \chi_v$ as usual. Let $q(x) \in \mathrm{H\ddot{o}l}_{\mathrm{loc}}(M_g)$ be such that*

$$A_1\big(r(x)\big) \leq q(x) \leq A_2\big(r(x)\big),$$

for some $A_j \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+_0)$, $j \in \{1,2\}$ *with* $A_2(r) \le \chi(r)$. *Suppose that there exists* $k \in (-\infty, 1]$ *for which*

$$\frac{A_j(r) - k\chi(r)}{\sqrt{\chi_{H_k^2 v}(r)}} \in L^1(+\infty) \qquad \text{for } j = 1, 2. \tag{222}$$

Let $\sigma > 1$, $B \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+_0)$ *and assume*

$$\frac{B(r)H_k(r)^{\sigma-1}}{\sqrt{\chi_{H_k^2 v}(r)}} \in L^1(+\infty). \tag{223}$$

Then, there exists a constant $\beta > 0$, *depending on* σ, g, A_j, B, k *such that the following holds: for each* $\Gamma_2 \in (0, \beta)$, *and for each* $b(x) \in \mathrm{H\ddot{o}l}_{\mathrm{loc}}(M_g)$ *satisfying*

$$|b(x)| \le B(r(x)) \qquad \text{on } M_g,$$

there exists $0 < \Gamma_1 \le \Gamma_2$ *such that the equation*

$$\Delta u + q(x)u - b(x)u^\sigma = 0 \tag{224}$$

possesses a positive solution $u \in \mathrm{H\ddot{o}l}^2_{\mathrm{loc}}(M_g)$ *with the property that, for some* $r_0 > 0$,

$$\Gamma_1 H_k(r(x)) \le u(x) \le \Gamma_2 H_k(r(x)) \qquad \text{for } r(x) \ge r_0. \tag{225}$$

If $q(x)$ *is radial, there exists* $\bar\beta \le \beta$ *such that, for each* $\gamma_\infty \in (0, \bar\beta)$, *there exists a solution of* (224) *with the further property*

$$\lim_{r(x)\to+\infty} \frac{u(x)}{H_k(r(x))} = \gamma_\infty. \tag{226}$$

Moreover, $\Gamma_2 \to 0$ *as* $\gamma_\infty \to 0$.

Remark 28. It is worth to remark that the general asymptotic relation (226) seems hardly obtainable from the proof of Theorem 10 on non-radial manifolds. In fact, for $k \neq 0$, the possibility of choosing a subsolution u_- and a supersolution u_+ sharing the same asymptotic behaviour at infinity, and also satisfying $u_- \le u_+$, strictly depends on having $G = \bar G$.

Remark 29. Particularizing Theorem 11 to the Yamabe problem on $\mathbb{H}^m_H$, and proceeding along the same lines as in the proof of Theorem 2, we conclude the further asymptotic relation described in Remark 4.

Remark 30. The above result enables us to justify the statements in Remark 8 in the Introduction. In fact, for $G(r) = 0$ and $k = 1$, by (261) in the Appendix, condition (222) reads

$$r \log r \big[A_j(r) - \chi(r)\big] \in L^1(+\infty)$$

and, by (260),

$$H_1(r) \sim C r^{-\frac{m-2}{2}} \log r \qquad \text{as } r \to +\infty.$$

Whence, (223) is equivalent to

$$B(r) r^{-\frac{m-2}{2}(\sigma-1)+1} (\log r)^\sigma \in L^1(+\infty).$$

A complementary non-existence result

The aim of this section is to prove the non-existence result contained in Theorem 12 below. Towards this aim we shall use the following simplified version of Lemma 3.2 of [30].

Lemma 5. *Let $M_{\tilde{g}}$ be a C^2-model of dimension m, $\tilde{v} = \tilde{g}^{m-1}$, and let $B(r) \in C^0(\mathbb{R}_0^+)$, $B > 0$ on $\mathbb{R}^+$ with*

$$i) \quad \sup_{\mathbb{R}_0^+} B < +\infty \quad , \qquad ii) \quad B(r) \geq \frac{C}{r^\mu} \tag{227}$$

for $r >> 1$, some $\mu \in [0, 2)$ and some constant $C > 0$. Suppose that

$$\liminf_{r \to +\infty} \frac{1}{r^{2-\mu}} \log \int_0^r \tilde{v}(t) \mathrm{d}t < +\infty. \tag{228}$$

Let γ be a C^2-solution of

$$\begin{cases} (\tilde{v}\gamma')' = \tilde{v} B \gamma^\sigma \\ \gamma(0) = \gamma_o > 0, \qquad \gamma'(0) = 0 \end{cases} \tag{229}$$

for some $\sigma > 1$, and let $[0, R)$ be its maximal interval of definition. Then $R < +\infty$, $\gamma' > 0$ on $(0, R)$ and

$$\gamma(r) \to +\infty \qquad \text{as } r \to R^-. \tag{230}$$

In the present case condition (227) $i)$ will not be necessary, and we briefly show how to overcome it.
Suppose $\sup_{\mathbb{R}_0^+} B = +\infty$ and that γ satisfies (229). Let $\tilde{B} \in C^0(\mathbb{R}_0^+)$, $\tilde{B} > 0$ satisfying both (227) $i), ii)$ and $\tilde{B}(r) \leq B(r)$. Then, by Lemma 5, if $\tilde{\gamma}$ is a solution of

$$\begin{cases} (\tilde{v}\tilde{\gamma}')' = \tilde{v} \tilde{B} \tilde{\gamma}^\sigma \\ \tilde{\gamma}(0) = \tilde{\gamma}_o > 0, \qquad \tilde{\gamma}'(0) = 0, \end{cases} \tag{231}$$

then $\tilde{\gamma}$ is positive and it explodes in finite time, say R. Suppose now that γ satisfies (229). Since $B > 0$, integrating (229) we see that $\gamma'(r) > 0$ for $r > 0$, hence $\gamma > 0$. This fact together with $\tilde{B}(r) \leq B(r)$ shows that γ is a subsolution of the equation in (231). Choose $\tilde{\gamma}_o < \gamma_o$ and let $u(x) = \gamma(r(x))$, $\tilde{u}(x) = \tilde{\gamma}(r(x))$. We then know that

$$\begin{cases} \Delta \tilde{u} = \tilde{B}(r(x)) \tilde{u}^\sigma \qquad \text{on } B_R \\ \tilde{u} = +\infty \qquad \text{on } \partial B_R \end{cases} \tag{232}$$

and

$$\Delta u \geq \tilde{B}(r(x)) u^\sigma \tag{233}$$

We can now conclude with the aid of a standard maximum principle technique. Suppose, by contradiction, that u is defined on all of M_g. Define $w = u - \tilde{u}$ and observe that, using (232) and (233),

$$\Delta w \geq \tilde{B}(r(x)) \left(u^\sigma - \tilde{u}^\sigma\right) \qquad \text{on } B_R. \tag{234}$$

Set

$$p(x) = \begin{cases} \sigma \tilde{u}(x)^{\sigma-1} \qquad \text{if } \tilde{u}(x) = u(x) \\ \dfrac{\sigma}{u(x) - \tilde{u}(x)} \displaystyle\int_{\tilde{u}(x)}^{u(x)} t^{\sigma-1} \mathrm{d}t \qquad \text{if } \tilde{u}(x) \neq u(x). \end{cases}$$

and note that $p \in C^0(B_R)$ and $p \geq 0$ on B_R. Inequality (234) can be rewritten as

$$\Delta w \geq \tilde{B}(r(x))p(x)w \qquad \text{on } B_R \tag{235}$$

with

$$\tilde{B}(r(x))p(x) \geq 0 \qquad \text{on } B_R. \tag{236}$$

Choose $\epsilon > 0$ so small that $w_{|\partial B_{R-\epsilon}} < 0$. Since $w(0) = \gamma_o - \tilde{\gamma}_o > 0$, w has an absolute maximum in $B_{R-\epsilon}$. Using (235), (236) and the maximum principle (see [29], [8]), this implies that w is constant, contradiction.

With the aid of our techniques, we can extend the above lemma to cover the Cauchy problem

$$\begin{cases} (v\gamma')' + Av\gamma = Bv\gamma^\sigma \\ \gamma(0) = \gamma_o > 0, \qquad \gamma'(0) = 0, \end{cases}$$

where the linear term A is subjected to the condition $A \leq \chi$.

Proposition 3. *Let M_g be a model with dimension $m \geq 3$ and radial sectional curvature satisfying $K_{\text{rad}} \leq 0$. Set $v = g^{m-1}$. Let*

$$\begin{aligned} &A(r) \in \text{Höl}_{\text{loc}}(\mathbb{R}_0^+), \qquad A(r) \leq k\chi(r) \quad \textit{on } \mathbb{R}^+ \\ &B(r) \in C^0(\mathbb{R}_0^+), \quad B > 0 \quad \textit{on } \mathbb{R}^+, \quad B(r) \geq \frac{C}{r^\mu H_k(r)^{\sigma-1}} \quad \textit{for } r >> 1, \end{aligned} \tag{237}$$

for some $\mu \in [0, 2)$, $k \in (-\infty, 1]$, $\sigma > 1$ and some constant $C > 0$. Assume

$$\liminf_{r\to+\infty} \frac{1}{r^{2-\mu}} \log \int_0^r \exp\left\{\frac{2}{m-2}\int_0^t \frac{g(s)}{g'(s)}A_-(s)\mathrm{d}s\right\} v(t)\mathrm{d}t < +\infty \tag{238}$$

Then any positive C^2-solution γ of

$$\begin{cases} (v\gamma')' + Av\gamma = Bv\gamma^\sigma \\ \gamma(0) = \gamma_o > 0, \qquad \gamma'(0) = 0 \end{cases} \tag{239}$$

is defined on a maximal interval $[0, R)$ with $R < +\infty$, and $\gamma(r) \to +\infty$ as $r \to R^-$.

Remark 31. Since $K_{\text{rad}} \leq 0$, M_g satisfies $g'' = (-K_{\text{rad}})g \geq 0$. Integrating twice, $g(r) \geq r$, which implies that v has property (V_{L1}) for each dimension $m \geq 3$ and $A \leq \chi$ is well defined.

Proof. Let h be the positive, C^2 solution of

$$\begin{cases} (vh')' + Avh = 0 \qquad \text{on } \mathbb{R}^+, \\ h(0) = 1, \qquad h'(0) = 0 \end{cases} \tag{240}$$

which is granted by $A \leq \chi$. Then, by Proposition 2, since $g'' \geq 0$ the function h satisfies the upper bound in (102). On $[0, R)$ we define $\xi = \gamma/h$. Then (239) and (101) imply that ξ satisfies

$$\begin{cases} \left(h^2 v\xi'\right)' = (Bh^{\sigma-1})h^2 v\xi^\sigma \qquad \text{on } [0, R) \\ \xi(0) = \gamma_o, \qquad \xi'(0) = 0. \end{cases} \tag{241}$$

We let $\tilde{g}(r) = h^{\frac{2}{m-1}}(r)g(r) \in C^2(\mathbb{R}^+)$. Then $\tilde{g}(0) = 0$, $\tilde{g}'(0) = 1$, whence $M_{\tilde{g}}$ is a model of class C^2. We want to apply Lemma 5 to $M_{\tilde{g}}$. Condition (228) is satisfied because of (102) and (238). As for condition (227), we need to have

$$B(r)h(r)^{\sigma-1} \geq \frac{C}{r^\mu} \qquad \text{for } r >> 1. \tag{242}$$

To show that this is the case we apply the usual Sturm comparison procedure. First, we observe that condition $A \leq \chi$ enables us to choose $A \leq \bar{A} \leq k\chi$ such that $\bar{A} = k\chi$ on $[r_1, +\infty)$, for some r_1 large. We then know that a solution $\bar{h}$ of

$$\begin{cases} (v\bar{h}')' + \bar{A}v\bar{h} = 0 & \text{on } \mathbb{R}^+, \\ \bar{h}(0) = 1, \qquad \bar{h}'(0) = 0 \end{cases}$$

satisfies $\bar{h}(r) \sim CH_k(r)$ as $r \to +\infty$. Now, by Sturm comparison $\bar{h} \leq h$, hence the assumption

$$B(r) \geq \frac{C}{r^\mu H_k(r)^{\sigma-1}}$$

implies the validity of (242). Concluding, by Lemma 5 it follows that ξ, and therefore γ, is defined on a finite maximal interval. Note that, since $z(r) \to +\infty$ as $r \to R^-$, then also $\gamma(r) \to +\infty$ as $r \to R^-$. $\square$

We are ready to state our non-existence result.

Theorem 12. *Let M_g be a model manifold of dimension $m \geq 3$ with $K_{\mathrm{rad}} \leq 0$. Set $v = g^{m-1}$ and $\chi = \chi_v$ as usual. Let $q(x), b(x) \in \mathrm{H\ddot{o}l}_{\mathrm{loc}}(M_g)$, and let A, B satisfy the assumptions*

$$\begin{aligned} &A(r) \in \mathrm{H\ddot{o}l}_{\mathrm{loc}}(\mathbb{R}^+_0), \qquad A(r) \leq k\chi(r) \quad \text{on } \mathbb{R}^+ \\ &B(r) \in C^0(\mathbb{R}^+_0), \quad B > 0 \quad \text{on } \mathbb{R}^+, \quad B(r) \geq \frac{C}{r^\mu H_k(r)^{\sigma-1}} \quad \text{for } r >> 1, \end{aligned} \tag{243}$$

for some $\mu \in [0, 2)$, $k \in (-\infty, 1]$, $\sigma > 1$ and some constant $C > 0$. Suppose that

$$q(x) \leq A(r(x)), \quad b(x) \geq B(r(x)) \qquad \text{for every } x \in M_g. \tag{244}$$

If

$$\liminf_{r\to+\infty} \frac{1}{r^{2-\mu}} \log \int_0^r \exp\left\{\frac{2}{m-2}\int_0^t \frac{g(s)}{g'(s)}A_-(s)\mathrm{d}s\right\} v(t)\mathrm{d}t < +\infty, \tag{245}$$

then the equation

$$\Delta u + q(x)u - b(x)u^\sigma = 0 \tag{246}$$

has no positive C^2-solution on M_g.

Proof. By contradiction suppose that u is a positive C^2-solution on M_g of (246). Fix $0 < \gamma_o < u(o)$. By Proposition 3, the solution γ of (239) explodes in finite time, say R. Then, the function $u_+(x) = \gamma(r(x))$ solves

$$\begin{aligned} \Delta u_+ + q(x)u_+ - b(x)u_+^\sigma &\leq \Delta u_+ + A(r)u_+ - B(r)u_+^\sigma \\ &= \gamma'' + \frac{v'}{v}\gamma' + A\gamma - B\gamma^\sigma = 0 \end{aligned} \tag{247}$$

and $u_+(x) \to +\infty$ as $x \to \partial B_R$. To conclude, define $\psi = u/u_+ \in C^2(B_R)$. Then a computation using (246) and (247) shows that ψ satisfies

$$\Delta\psi + 2 < \nabla \log u_+, \nabla\psi > \geq b(x)\left(u^{\sigma-1} - u_+^{\sigma-1}\right)\psi \qquad \text{on } B_R. \tag{248}$$

Let $\Omega = \{x \in B_R : \psi(x) > 1\}$. Then, Ω is open, $o \in \Omega \neq \emptyset$, and $\overline{\Omega} \Subset B_R$ since $\psi \to 0$ on ∂B_R. Hence ψ has an absolute maximum in Ω. However, from (248) and $b(x) \geq 0$ on Ω we have

$$\Delta\psi + 2 < \nabla \log u_+, \nabla\psi > \geq 0 \qquad \text{on } \Omega,$$

from which we obtain a contradiction with the aid of the maximum principle (see [29], [8]). ☐

We conclude by specializing the last theorem in the case of the Yamabe problem for the hyperbolic space $\mathbb{H}^m_H$. Next result gives a condition mildly more demanding than (26) for the non-existence of conformal deformations, which shows the sharpness of Theorem 2.

Corollary 6. *Consider the hyperbolic space $\mathbb{H}^m_H$ of dimension $m \geq 3$. If $\widetilde{s}(x) \in C^\infty(\mathbb{H}^m_H)$ satisfies*

$$\widetilde{s}(x) \leq 0, \qquad \textit{and} \qquad \widetilde{s}(x) \leq -C\frac{e^{2Hr(x)}}{r(x)} \qquad \textit{outside some ball,}$$

for some $C > 0$, then the Poincaré metric cannot be conformally deformed to a new metric of scalar curvature $\widetilde{s}(x)$.

Proof. We realize $\mathbb{H}^m_H$ as a model with $g(r) = H^{-1}\sinh(Hr)$. Then, g satisfies $g'' \geq 0$ and (V_{L1}). We remark that the conformal factor u has to solve (216). Define

$$\sigma = \frac{m+2}{m-2}, \qquad A(r) = \frac{H^2 m(m-2)}{4} < k\chi(r) \quad \text{for} \quad k = \frac{m(m-2)}{(m-1)^2} < 1,$$

the inequality being a consequence of (56), and

$$b(x) = -\frac{\widetilde{s}(x)}{c_m}, \qquad B(r) \geq 0, \qquad B(r) = C\frac{e^{2Hr(x)}}{r(x)} \qquad \text{for } r \geq 1.$$

Then, we have by (256) in the Appendix

$$H_k(r) \sim Ce^{-\frac{H(m-2)}{2}r}, \qquad \text{so that } H_k(r)^{\sigma-1} \sim Ce^{-2Hr},$$

and, by our assumption on $\widetilde{s}(x)$,

$$-\frac{\widetilde{s}(x)}{c_m} \geq B(r(x)) = C\frac{e^{2Hr(x)}}{r(x)} \geq \frac{C}{r(x)H_k\big(r(x)\big)^{\sigma-1}}$$

for $r(x) >> 1$, up to changing C appropriately. Condition (238) is satisfied whenever $\mu \in [0,1]$. The choice $\mu = 1$ is allowed, and applying Theorem 12 for $M_g = \mathbb{H}^m_H$ we conclude. ☐

Appendix: explicit computations

In this appendix, we perform explicit computations of g, χ, H_k and $\chi_{H_k^2 v}$ for the two relevant classes of G described in the Introduction, that is,

$$\begin{array}{llll} (i) & m \geq 2 & G(r) = H^2(1+r^2)^{\alpha/2} & \text{for} \quad H > 0, \ \alpha \geq -2 \\ (ii) & m \geq 3 & G(r) = -\dfrac{H^2}{(1+r)^2} & \text{for} \quad H \in [0, 1/2]\,. \end{array} \tag{249}$$

First, we produce a positive solution g of

$$\begin{cases} g'' - Gg \leq 0 & \text{on } \mathbb{R}^+ \\ g(0) = 0, \quad g'(0) = 1 \end{cases} \tag{250}$$

satisfying $v^{-1} = g^{1-m} \in L^1(+\infty)$. Then we find, when possible, a closed expression for the critical curve χ. Moreover, we compute the asymptotics for H_k and $\chi_{H_k^2 v}$ as $r \to +\infty$, up to some unessential constant. For $\chi_{H_k^2 v}$, we can avail of formulas (59). In this way, the requirements

$$A_j(r) \leq \chi(r) \quad \text{on } \mathbb{R}^+, \qquad \frac{A_j(r) - k\chi(r)}{\sqrt{\chi_{H_k^2 v}(r)}} \in L^1(+\infty), \qquad \frac{B(r) H_k(r)^{\sigma-1}}{\sqrt{\chi_{H_k^2 v}(r)}} \in L^1(+\infty)$$

of the main Theorems 9, 10 will become entirely explicit. Note that, to check the first integrability above, we only need a Taylor expansion of χ in a neighbourhood of $+\infty$, that can be performed, with some effort, also for the most difficult cases $\alpha > 0$ and $\alpha \in (-2, 0]$ in Class (i).

As a matter of fact, in some cases the initial conditions $g(0) = 0$, $g'(0) = 1$ needed to ensure that the model M_g is C^2 at the origin prevent appropriate choices that simplify computations. However, it is worth to observe that, in order to apply the Laplacian comparison theorem, it is enough that g solves

$$\begin{cases} g'' - Gg \leq 0 & \text{on } \mathbb{R}^+ \\ \frac{g'}{g} \leq \frac{1}{r} + o(1) & \text{as } r \to 0^+. \end{cases} \tag{251}$$

In some of the cases presented below, g will satisfy (251) instead of the stronger (250). However, this does not affect the validity of the conclusions of Theorem 9. In what follows, with the symbol C we mean a constant whose value may change from line to line.

Class (i): $G(r) = H^2(1+r^2)^{\alpha/2}$, $m \geq 2$.

The case $\alpha > 0$:

We choose

$$g(r) = r^{\frac{1}{2}} I_{\frac{1}{2+\alpha}}\left(\frac{2H}{2+\alpha} r^{1+\frac{\alpha}{2}}\right),$$

where $I_\nu(s)$ is the modified Bessel function of order ν (see [16], p.102). Note that g is a solution of the singular equation $g'' = H^2 r^\alpha g$ on $\mathbb{R}^+$, with initial conditions $g(0) = 0$,

$g'(0) = C > 0$ for some constant C. Hence, g solves (251) since $\alpha > 0$. Using the asymptotic relation

$$I_\nu(s) = \frac{e^s}{\sqrt{2\pi s}}(1 + o(1)) \qquad \text{as} \quad s \to +\infty \tag{252}$$

(see [16], p.123) we deduce that v satisfies (V_{L1}) and

$$\begin{aligned} v(r) \quad &\sim \quad Cr^{-\frac{(m-1)\alpha}{4}} \exp\left(\tfrac{2H(m-1)}{2+\alpha} r^{1+\frac{\alpha}{2}}\right), \\ \int_r^{+\infty} \frac{\mathrm{d}s}{v(s)} \quad &\sim \quad C^{-1}\tfrac{1}{H(m-1)} r^{(m-3)\frac{\alpha}{4}} \exp\left(-\tfrac{2H(m-1)}{2+\alpha} r^{1+\frac{\alpha}{2}}\right), \end{aligned} \tag{253}$$

for some $C > 0$. Consequently,

$$\begin{cases} H_1(r) \sim Cr^{1+(m+1)\frac{\alpha}{8}} \exp\left\{-\frac{H(m-1)}{2+\alpha} r^{1+\frac{\alpha}{2}}\right\} \\ H_k(r) \sim Cr^{(m-3)\frac{\alpha}{8}(1-\sqrt{1-k})} \exp\left\{-\frac{H(m-1)(1-\sqrt{1-k})}{2+\alpha} r^{1+\frac{\alpha}{2}}\right\} \qquad \text{for } k < 1, \end{cases}$$

and with some computations

$$\begin{cases} \chi_{H_1^2 v}(r) \sim \frac{(\alpha+2)^2}{16r^2} \\ \chi_{H_k^2 v}(r) \sim \frac{(m-1)^2H^2(1-k)}{4} r^\alpha \qquad \text{for } k < 1. \end{cases} \tag{254}$$

The case $\alpha \in (-2, 0]$:

A solution of (250) is given by

$$g(r) = H^{-1}\sinh\left(\frac{2H}{2+\alpha}\left[(1+r)^{1+\frac{\alpha}{2}} - 1\right]\right),$$

and $v^{-1} \in L^1(+\infty)$. A closed expression of χ cannot be computed except for the case $\alpha = 0$, which characterizes the hyperbolic space, for which we refer the reader to Example 1. However, from

$$\begin{aligned} v(r) \quad &\sim \quad C\exp\left(\tfrac{2H(m-1)}{2+\alpha} r^{1+\frac{\alpha}{2}}\right), \\ \int_r^{+\infty} \frac{\mathrm{d}s}{v(s)} \quad &\sim \quad C^{-1}\tfrac{1}{H(m-1)} r^{-\alpha/2} \exp\left(-\tfrac{2H(m-1)}{2+\alpha} r^{1+\frac{\alpha}{2}}\right), \end{aligned} \tag{255}$$

for some $C > 0$ (note that here we have used $\alpha \in (-2, 0]$ to replace $(1 + r)$ with r), we deduce

$$\begin{cases} H_1(r) \sim Cr^{1+\frac{\alpha}{4}} \exp\left\{-\frac{H(m-1)}{2+\alpha} r^{1+\frac{\alpha}{2}}\right\} \\ H_k(r) \sim Cr^{-\frac{\alpha}{4}(1-\sqrt{1-k})} \exp\left(-\frac{H(m-1)(1-\sqrt{1-k})}{2+\alpha} r^{1+\frac{\alpha}{2}}\right) \qquad \text{for } k < 1 \end{cases} \tag{256}$$

thus

$$\begin{cases} \chi_{H_1^2 v}(r) \sim \frac{(\alpha+2)^2}{16r^2} \\ \chi_{H_k^2 v}(r) \sim \frac{(m-1)^2H^2(1-k)}{4} r^\alpha \qquad \text{for } k < 1. \end{cases} \tag{257}$$

The case $\alpha = -2$:

Define

$$H' = \frac{1}{2}(1 + \sqrt{1 + 4H^2}) \quad \in (1, +\infty),$$

and consider $g(r) = (1 + r)^{H'}$. Then, g solves (251) and

$$\chi(r) = \frac{(H'(m-1) - 1)^2}{4(1+r)^2}.$$

Furthermore,

$$\begin{cases} H_1(r) \sim Cr^{-\frac{(m-1)H'-1}{2}} \log r \\ H_k(r) \sim Cr^{-\frac{\left[(m-1)H'-1\right](1-\sqrt{1-k})}{2}} \qquad \text{for } k < 1, \end{cases}$$

from which

$$\begin{cases} \chi_{H_1^2 v}(r) \sim \frac{1}{4r^2 \log^2 r} \\ \chi_{H_k^2 v}(r) \sim \frac{\left[(m-1)H'-1\right]^2 (1-k)}{4r^2} \qquad \text{for } k < 1. \end{cases} \tag{258}$$

Class (ii): $G(r) = -H^2/(1+r)^2$, $m \geq 3$.

In this case, a model M_g with radial sectional curvature $K_{\mathrm{rad}} = -G(r) \geq 0$ and dimension $m = 2$ is necessary parabolic. In fact, by Bishop-Gromov volume comparison theorem the volume of a geodesic ball B_r centered at the origin grows at most quadratically as a function of r, thus parabolicity follows by a criterion in [11]. Therefore, the restriction $m \geq 3$ is necessary to fulfil the non-parabolicity assumption $g^{1-m} \in L^1(+\infty)$. Define

$$H'' = \frac{1 + \sqrt{1 - 4H^2}}{2} \quad \in \left[\frac{1}{2}, 1\right]. \tag{259}$$

The case $H \in [0, 1/2)$:

An explicit solution g of (250) with the equality sign is

$$g(r) = \frac{1}{\sqrt{1 - 4H^2}} \Big((1+r)^{H''} - (1+r)^{1-H''} \Big)$$

(see [35], p.45). If $H = 0$, $g(r) = r$ and so

$$\chi(r) = \frac{(m-2)^2}{4r^2}.$$

However, for H positive the algebraic integration of $1/v$ seems complicate, so we prefer choosing $g(r) = r^{H''}$. Since $H'' \leq 1$, g satisfies (251) and

$$\chi(r) = \frac{\big(H''(m-1) - 1\big)^2}{4r^2}.$$

In both cases $H = 0$ and $H > 0$,

$$\begin{cases} H_1(r) \sim Cr^{-\frac{(m-1)H''-1}{2}} \log r \\ H_k(r) \sim Cr^{-\frac{\left[(m-1)H''-1\right](1-\sqrt{1-k})}{2}} \qquad \text{for } k < 1, \end{cases} \tag{260}$$

so that

$$\begin{cases} \chi_{H_1^2 v}(r) \sim \frac{1}{4r^2 \log^2 r} \\ \chi_{H_k^2 v}(r) \sim \frac{\left[(m-1)H''-1\right]^2(1-k)}{4r^2} \qquad \text{for } k<1. \end{cases} \tag{261}$$

The case $H = 1/2$:

An explicit solution of (250) is

$$g(r) = \sqrt{1+r}\log(1+r), \tag{262}$$

and g satisfies $v^{-1} = g^{1-m} \in L^1(+\infty)$ for each $m \geq 3$. We divide into two subcases, according to whether $m = 3$ or $m \geq 4$.

- If $m = 3$, a closed expression of χ can be computed:

$$\chi(r) = \frac{1}{4(1+r)^2\log^2(1+r)}.$$

Moreover,

$$\begin{cases} H_1(r) \sim C\log^{-\frac{1}{2}} r \log\log r \\ H_k(r) \sim C\log r^{-\frac{1-\sqrt{1-k}}{2}} \qquad \text{for } k<1, \end{cases}$$

so that

$$\begin{cases} \chi_{H_1^2 v}(r) \sim \frac{1}{4r^2\log^2 r\log^2(\log r)} \\ \chi_{H_k^2 v}(r) \sim \frac{1-k}{4r^2\log^2 r} \qquad \text{for } k<1. \end{cases}$$

- If $m \geq 4$, the choice of g in (262) does not allow a straightforward computation of χ. For this reason, we prefer using again $g(r) = r^{H''} = \sqrt{r}$. Then, g solves (251) and, since $m \geq 4$, $v^{-1} = g^{1-m} \in L^1(+\infty)$. The critical curve is given by

$$\chi(r) = \frac{(m-3)^2}{16r^2},$$

the asymptotics for $H_k(r)$ read

$$\begin{cases} H_1(r) \sim Cr^{-\frac{m-3}{4}}\log r \\ H_k(r) \sim Cr^{-\frac{(m-3)(1-\sqrt{1-k})}{4}} \qquad \text{for } k<1, \end{cases}$$

and finally

$$\begin{cases} \chi_{H_1^2 v}(r) \sim \frac{1}{4r^2\log^2 r} \\ \chi_{H_k^2 v}(r) \sim \frac{(m-3)^2(1-k)}{4r^2} \qquad \text{for } k<1. \end{cases}$$

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